Handwritten notes and problems

by Prof. M. L. Aggarwal

Topic - Functions

Functions

1. If $f(x) = \frac{x-1}{x+1}$, then $f(ax)$ in terms of $f(x)$ is equal to

(a) $\frac{f(x)+a}{1+a\,f(x)}$ (b) $\frac{(a-1)f(x)+a+1}{(a+1)f(x)+a-1}$

✓(c) $\frac{(a+1)f(x)+a-1}{(a-1)f(x)+a+1}$ (d) none of these UD

2. The range of the function $f(x) = \sqrt{2-x} + \sqrt{x+1}$ is

(a) $(\sqrt{3}, \sqrt{6})$ (b) $[0, \sqrt{6}]$ ✓(c) $[\sqrt{3}, \sqrt{6}]$ (d) none of these UD

3. If two functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are defined as follows:

$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$ and $g(x) = \begin{cases} -1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

then $(g \circ f)(e) + (f \circ g)(\pi)$ is UD

✓(a) -1 (b) 0 (c) 1 (d) 2

4. If $f(x) = 1 + x^{1/3}$ and $g(f(x)) = 3 - x^{1/3} + x$, then $g(5)$ is equal to

(a) 27 (b) 36 ✓(c) 63 (d) 75 UD

5. If $f(x) = \frac{a^x}{a^x + \sqrt{a}}$ $(a > 0)$, then the value of $\sum_{r=1}^{2n-1} 2\, f\left(\frac{r}{2n}\right)$ is

✓(a) $2n-1$ (b) $n-1$ (c) $2n+1$ (d) none of these UD

6. If $[x]$ and $\{x\}$ represent integral and fractional part of x, then the expression $[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000}$ is equal to UD

(a) $\frac{2001x}{2}$ (b) $x + 2001$ ✓(c) x (d) $[x] + \frac{2001}{2}$

7. If $[x]$ denotes the greatest integer less than or equal to x, then the domain of the real valued function $\frac{\log|x^2 - x + 2|}{[x+\frac{1}{2}]}$ is UD

(a) $\left[\frac{3}{2}, \infty\right)$ (b) $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$ (c) $\left(\frac{1}{2}, 2\right) \cup (2, \infty)$ (d) none of these

8. The domain of the function $\frac{1}{\sqrt{[x]^2 - [x] - 6}}$ is (NCERT Exemplar)

(a) $(-\infty, -2) \cup (3, \infty)$ ✓(b) $(-\infty, -2) \cup [4, \infty)$ (c) $(-\infty, -1] \cup [4, \infty)$ (d) none of these

9. If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are two functions such that $f(x) = 2x - 3$ and $g(x) = x^3 + 5$, then $(f \circ g)^{-1}(x)$ is equal to

(a) $\left(\frac{x+7}{2}\right)^{1/3}$ (b) $\left(x - \frac{7}{2}\right)^{1/3}$ (c) $\left(\frac{x-2}{7}\right)^{1/3}$ ✓(d) $\left(\frac{x-7}{2}\right)^{1/3}$

10. If $f(x) = \dfrac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}$, then

(a) f is an even function ✓(b) f is an odd function

✓(c) $D_f = \mathbb{R} - \{0\}$ ✓(d) $\lim\limits_{x \to 0} f(x)$ does not exist

11. The function $f : A \to B$ defined by $f(x) = -x^2 + 6x - 8$ is a bijection if

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✓(a) $A = [3, \infty)$ and $B = (-\infty, 1]$ ✓(b) $A = (-\infty, 3]$ and $B = (-\infty, 1]$

(c) $A = (-\infty, 3]$ and $B = [1, \infty)$ (d) $A = [3, \infty)$ and $B = [1, \infty)$

12. The range of the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = x^3 - x^2 + 4x + 2\sin^{-1} x \text{ is}$$

UD

(a) $[-\pi - 2, 0]$ (b) $[2, 3]$ (c) $[0, 4+\pi]$ (d) none of these

13. If $f : \mathbb{R} \to \mathbb{R}$ is a function defined by $f(x) = x^3 + x^2 + 3x + \sin x$, then f is

UD

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

14. The function $f : (-\infty, -1] \to (0, e^5]$ defined by

$$f(x) = e^{x^3 - 3x + 2} \text{ is}$$

UD

(a) one-one onto (b) one-one into

(c) many-one into (d) many-one onto

* Let a real function $f : D_f \to R_f$ be invertible, then

$f^{-1} : R_f \to D_f$ is given by $f^{-1}(y) = x$ iff $y = f(x)$ for all $x \in D_f$ and $y \in R_f$. Thus, the roles of x and y are just interchanged during the transition from f to f^{-1}.

Infact, graph of $f = \{(x, y) : y = f(x), \text{ for all } x \in D_f\}$ and

$$\text{graph of } f^{-1} = \{(y, x) : x = f^{-1}(y), \text{ for all } y \in R_f\}$$
$$= \{(y, x) : y = f(x), \text{ for all } x \in D_f\}.$$

Thus, $(x, y) \in$ graph of f iff $(y, x) \in$ graph of f^{-1}.

The point (y, x) is the reflection of the point (x, y) in the line $y = x$.

Therefore, the graph of f^{-1} can be obtained from the graph of f by reflecting it through the line $y = x$.

Solutions — Functions

1. $f(x) = \frac{x-1}{x+1} \Rightarrow x f(x) + f(x) = x - 1 \Rightarrow x = \frac{1+f(x)}{1-f(x)}$.

Now $f(ax) = \frac{ax-1}{ax+1} = \frac{a \cdot \frac{1+f(x)}{1-f(x)} - 1}{a \cdot \frac{1+f(x)}{1-f(x)} + 1} = \frac{a(1+f(x)) - 1 + f(x)}{a(1+f(x)) + 1 - f(x)}$

$= \frac{(a+1) f(x) + a - 1}{(a-1) f(x) + a + 1}$.

2. Let $y = f(x) = \sqrt{2-x} + \sqrt{x+1}$

i.e. $y = \sqrt{2-x} + \sqrt{x+1}$, $y \geq 0$

$\Rightarrow y^2 = (2-x) + (x+1) + 2\sqrt{(2-x)(x+1)}$

$= 3 + 2\sqrt{-x^2+x+2} = 3 + 2\sqrt{\frac{9}{4} - (x^2 - x + \frac{1}{4})}$

$= 3 + 2\sqrt{(\frac{3}{2})^2 - (x - \frac{1}{2})^2}$

Domain of $\sqrt{2-x} = (-\infty, 2]$, domain of $\sqrt{x+1} = [-1, \infty)$

$D_f = (-\infty, 2] \cap [-1, \infty) = [-1, 2]$

f is continuous in $[-1, 2]$

Minimum value of $y^2 = 3$ when $x = 2$ or $x = -1$

Maximum value of $y^2 = 3 + 2\sqrt{(\frac{3}{2})^2 - 0}$ when $x = \frac{1}{2}$

$= 3 + 2 \cdot \frac{3}{2} = 6$.

$\therefore$ Minimum value of $y = \sqrt{3}$ and maximum value of $y = \sqrt{6}$

$\therefore$ Range of the function $f = [\sqrt{3}, \sqrt{6}]$

3. $(g \circ f)(e) + (f \circ g)\pi = g(f(e)) + f(g(\pi))$ | e and π are irrational numbers

$= g(1) + f(0) = -1 + 0 = -1$

4. $f(x) = 1 + x^{1/3} \Rightarrow x = (f(x) - 1)^3$

• $g(f(x)) = 3 - x^{1/3} + x = 3 - ((f(x)-1)^3)^{1/3} + (f(x)-1)^3$

$= 3 - (f(x) - 1) + (f(x) - 1)^3$

(To find $g(5)$, we put $f(x) = 5$)

$\therefore g(5) = 3 - (5-1) + (5-1)^3 = 3 - 4 + 64 = 63$

5. Given $f(x) = \frac{a^x}{a^x + \sqrt{a}}$ $(a > 0)$

$\Rightarrow f(1-x) = \frac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \frac{a \cdot a^{-x}}{a \cdot a^{-x} + \sqrt{a}} = \frac{\sqrt{a}}{\sqrt{a} + a^x}$

$\therefore f(x) + f(1-x) = \frac{a^x}{a^x + \sqrt{a}} + \frac{\sqrt{a}}{\sqrt{a} + a^x} = 1$. ---- (i)

Thus, $f(x) + f(1-x) = 1$ for all $x \in \mathbb{R}$

$$\therefore \sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right) = 2\left[f\left(\frac{1}{2n}\right)+f\left(\frac{2}{2n}\right)+f\left(\frac{3}{2n}\right)+\cdots+f\left(\frac{2n-2}{2n}\right)+f\left(\frac{2n-1}{2n}\right)\right]$$

$$= 2\left[\left(f\left(\frac{1}{2n}\right)+f\left(\frac{2n-1}{2n}\right)\right)+\left(f\left(\frac{2}{2n}\right)+f\left(\frac{2n-2}{2n}\right)\right)+\cdots\right.$$

Now $f\left(\frac{1}{2n}\right)+f\left(\frac{2n-1}{2n}\right) = f\left(\frac{1}{2n}\right)+f\left(1-\frac{1}{2n}\right) = 1$ (using (i))

Thus the sum of two terms equidistant from the beginning and end are equal to 1 (each)

Middle term $= f\left(\frac{n}{2n}\right) = f\left(\frac{1}{2}\right) = \dfrac{a^{1/2}}{a^{1/2}+\sqrt{a}} = \dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{a}} = \dfrac{1}{2}$.

$$\therefore \sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right) = 2\left[1+1+1+\cdots \text{ to } (n-1) \text{ terms } + \frac{1}{2}\right]$$

$$= 2\left(n-1+\frac{1}{2}\right) = 2\left(n-\frac{1}{2}\right) = 2n-1.$$

6. $x = [x]+\{x\} \Rightarrow \{x\} = x-[x]$

$\therefore \{x+1\} = (x+1)-[x+1] = (x+1)-([x]+1) = x-[x]$.

Thus $\{x+h\} = \{x\}$, for all $h \in I$.

$$[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000} = [x] + \frac{\{x+1\}+\{x+2\}+\cdots+\{x+2000\}}{2000}$$

$$= [x] + \frac{\{x\}+\{x\}+\{x\}+\cdots \text{ 2000 terms}}{2000} = [x]+\{x\} = x.$$

7. Let $f(x) = \log_{[x+\frac{1}{2}]} |x^2-x-2|$,

For D_f, $x^2-x-2 \neq 0 \Rightarrow (x-2)(x+1) \neq 0 \Rightarrow x \neq 2, -1$.

Also $[x+\frac{1}{2}] > 0$ and $[x+\frac{1}{2}] \neq 1 \Rightarrow [x+\frac{1}{2}] \geq 2$

$\Rightarrow x+\frac{1}{2} \geq 2 \Rightarrow x \geq \frac{3}{2}$ but $x \neq 2$

$\Rightarrow D_f \in \left[\frac{3}{2}, 2\right) \cup (2, \infty)$.

8. Let $f(x) = \dfrac{1}{\sqrt{[x]^2-[x]-6}}$

For D_f, $[x]^2-[x]-6 > 0 \Rightarrow ([x]+2)([x]-3) > 0$

$\Rightarrow [x] < -2$ or $[x] > 3 \Rightarrow x < -2$ or $x \geq 4$

$\Rightarrow D_f = (-\infty, -2) \cup [4, \infty)$

9. $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two functions such that

$f(x) = 2x - 3$ and $g(x) = x^3 + 5$, $f\circ g$ exists and $f\circ g = \mathbb{R}$
$f\circ g : \mathbb{R} \to \mathbb{R}$

~~Note that~~ $(f\circ g)(x) = f(g(x)) = f(x^3+5) = 2(x^3+5) - 3 = 2x^3 + 7$

<u>$f\circ g$ is one-one</u>

Let $x_1, x_2 \in \mathbb{R}$ be such that $f(x_1) = f(x_2) \Rightarrow 2x_1^3 + 7 = 2x_2^3 + 7$

$\Rightarrow x_1^3 - x_2^3 = 0 \Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \Rightarrow x_1 = x_2$

$(\because x_1^2 + x_1x_2 + x_2^2 = (x_1 + \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 > 0$ for all $x_1, x_2 \in \mathbb{R}$ except when $x_1 = x_2 = 0)$

$\Rightarrow f$ is one-one.

<u>f is onto.</u>

consider any $y \in \mathbb{R}$ (codomain of $f\circ g$).

Then $(f\circ g)(x) = y \Rightarrow 2x^3 + 7 = y \Rightarrow x = \left(\frac{y-7}{2}\right)^{1/3} \in \mathbb{R}$

Thus, for every $y \in \mathbb{R}$ (codomain of $f\circ g$), there exists $x = \left(\frac{y-7}{2}\right)^{1/3} \in \mathbb{R}$ (domain of $f\circ g$) such that

$$(f\circ g)(x) = (f\circ g)\left(\left(\frac{y-7}{2}\right)^{1/3}\right) = 2\left(\left(\frac{y-7}{2}\right)^{1/3}\right)^3 + 7$$
$$= 2\left(\frac{y-7}{2}\right) + 7 = y$$

$\Rightarrow$ every element in the codomain of $f\circ g$ has its preimage in the domain of $f\circ g$

$\Rightarrow f\circ g$ is onto.

Hence, $f\circ g$ is bijective, so its inverse exists.

As $(f\circ g)(x) = y \Rightarrow (f\circ g)^{-1}(y) = x \Rightarrow (f\circ g)^{-1}(y) = \left(\frac{y-7}{2}\right)^{1/3}$

i.e. $(f\circ g)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$.

10. $f(x) = \dfrac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x} = \dfrac{|\sin x|}{x}$, $D_f = \mathbb{R} - \{0\}$

$f(-x) = \dfrac{|\sin(-x)|}{-x} = \dfrac{|-\sin x|}{-x} = -\dfrac{|\sin x|}{x} = -f(x)$, $\forall x \in D_f$

$\Rightarrow f$ is an odd function.

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{|\sin x|}{x}$ $\quad (x \to 0^+ \Rightarrow x \in (0, \frac{\pi}{2}) \Rightarrow \sin x > 0 \Rightarrow |\sin x| = \sin x)$

$= \lim\limits_{x\to0^+} \dfrac{\sin x}{x} = 1$

and $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{|\sin x|}{x}$ $\quad (x \to 0^- \Rightarrow \sin x < 0 \Rightarrow |\sin x| = -\sin x)$

$= \lim\limits_{x\to0^-} -\dfrac{\sin x}{x} = -1$

$\Rightarrow \lim\limits_{x\to0^+} f(x) \neq \lim\limits_{x\to0^+} f(x) \Rightarrow \lim\limits_{x\to0} f(x)$ does not exist.

11. $f(x) = -x^2 + 6x - 8 = -(x^2 - 6x + 9) + 1$

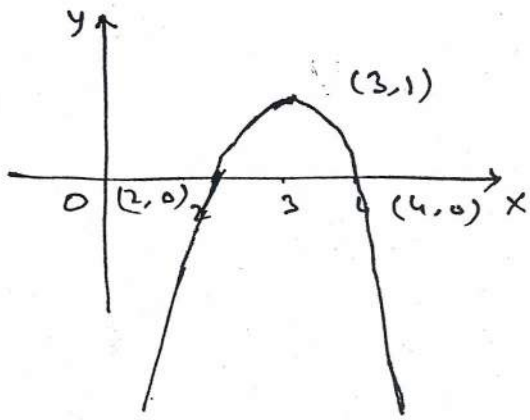
$\Rightarrow y = -(x-3)^2 + 1 \Rightarrow y - 1 = -(x-3)^2$,

which represents a downward parabola with vertex $(3,1)$

It meets x-axis i.e. $y = 0$ when $-x^2 + 6x - 8 = 0$

$\Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0 \Rightarrow x = 2, 4$

i.e. at point $(2,0)$, $(4,0)$.



$D_f = \mathbb{R}$

For R_f, $(x-3)^2 \geq 0$

$\Rightarrow -(x-3)^2 \leq 0$

$\Rightarrow -(x-3)^2 + 1 \leq 1$

$\Rightarrow y \leq 1$

$\therefore R_f = (-\infty, 1]$

(a) If $A = (-\infty, 3]$ and $B = (-\infty, 1]$, then f is one-one onto

(b) If $A = [3, \infty)$ and $B = (-\infty, 1]$, then f is one-one onto.

12. $f(x) = x^3 - x^2 + 4x + 2\sin^{-1} x$

$\therefore f'(x) = 3x^2 - 2x + 4 + \dfrac{2}{\sqrt{1-x^2}}$

$$\left[\begin{array}{l} f(0) = 0 \\ f(1) = 1 - 1 + 4 + 2\sin^{-1} 1 \\ \quad\;\; = 4 + 2 \cdot \frac{\pi}{2} = 4 + \pi \end{array}\right.$$

$= 3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{3} + 4 + \dfrac{2}{\sqrt{1-x^2}}$

$= 3\left(x - \frac{1}{3}\right)^2 + \frac{11}{3} + \dfrac{2}{\sqrt{1-x^2}} > 0$ for all $x \in [0, 1)$

$\Rightarrow f$ is increasing (strictly) in $[0,1]$

$\therefore R_f = [f(0), f(1)] = [0, 4+\pi]$

13. $f(x) = x^3 + x^2 + 3x + \sin x$

$f'(x) = 3x^2 + 2x + 3 + \cos x = 3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{9} + 3 + \cos x$

$= 3\left(x + \frac{1}{3}\right)^2 + \frac{8}{3} + \cos x > 0$ for all $x \in \mathbb{R}$

$\left(3\left(x + \frac{1}{3}\right)^2 \geq 0 \text{ and } -1 \leq \cos x \leq 1 \Rightarrow f'(x) > 0\right)$

$\Rightarrow f$ is strictly increasing $\Rightarrow f$ is one-one.

Also range of $f = \mathbb{R} \Rightarrow f$ is onto

$\therefore f$ is one-one onto.

14. $f(x) = e^{x^3-3x+2}$

$f'(x) = e^{x^3-3x+2} \cdot (3x^2-3) = 3e^{x^3-3x+2}(x+1)(x-1)$

Now, $f'(x) > 0 \Rightarrow (x+1)(x-1) > 0 \Rightarrow x < -1$ or $x > 1$

$\therefore$ f is strictly increasing in $(-\infty, -1] \cup [1, \infty)$

but $D_f = (-\infty, -1]$

$\Rightarrow$ f is strictly increasing in $(-\infty, -1]$ i.e. in its entire domain $\Rightarrow$ f is one-one.

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} e^{x^3-3x+2} \to 0$

and $f(-1) = e^{-1+3+2} = e^4$

which is a proper subset of $(0, e^5]$

$\therefore$ $R_f = (0, e^4]$

$\Rightarrow$ f is into.

Hence, function f is one-one into.