Handwritten notes and problems

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Topic - Inverse Trigonometric Functions

Inverse Trigonometric Functions

Function	Domain	Range (Principal Values)
$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0,\pi)$
$\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
$\operatorname{cosec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

Properties of Inverse Trigonometric functions

1. (i) $\sin(\sin^{-1}x) = x,\ |x|\le 1$ (ii) $\cos(\cos^{-1}x) = x,\ |x|\le 1$

(iii) $\tan(\tan^{-1}x) = x,\ x\in R$ (iv) $\cot(\cot^{-1}x) = x,\ x\in R$

(v) $\sec(\sec^{-1}x) = x,\ |x|\ge 1$ (vi) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x,\ |x|\ge 1$

2. (i) $\sin^{-1}(\sin x) = x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (ii) $\cos^{-1}(\cos x) = x,\ x\in[0,\pi]$

(iii) $\tan^{-1}(\tan x) = x,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (iv) $\cot^{-1}(\cot x) = x,\ x\in(0,\pi)$

(v) $\sec^{-1}(\sec x) = x,\ x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}x) = x,\ x\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

3. (i) $\sin^{-1}(-x) = -\sin^{-1}x,\ |x|\le 1$ (ii) $\cos^{-1}(-x) = \pi-\cos^{-1}x,\ |x|\le 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x,\ x\in R$ (iv) $\cot^{-1}(-x) = \pi-\cot^{-1}x,\ x\in R$

(v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x,\ |x|\ge 1$ (vi) $\sec^{-1}(-x) = \pi-\sec^{-1}x,\ |x|\ge 1$

4. (i) $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2},\ |x|\le 1$

(ii) $\tan^{-1}x+\cot^{-1}x = \frac{\pi}{2},\ x\in R$

(iii) $\sec^{-1}x+\operatorname{cosec}^{-1}x = \frac{\pi}{2},\ |x|\ge 1$

5. (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x},\ |x|\ge 1$ (ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x},\ |x|\ge 1$

(iii) $\cot^{-1}x = \begin{cases}\tan^{-1}\frac{1}{x}, & x>0\\ \pi+\tan^{-1}\frac{1}{x}, & x<0\end{cases}$

6. (i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2},\ 0\le x\le 1$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2},\ 0\le x\le 1$

(iii) $\cos(\sin^{-1}x) = \sin(\cos^{-1}x) = \sqrt{1-x^2},\ |x|\le 1$

7. (i) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\dfrac{x+y}{1-xy} & \text{if } xy<1\\ \pi+\tan^{-1}\dfrac{x+y}{1-xy} & \text{if } x>0, y>0, xy>1\\ -\pi+\tan^{-1}\dfrac{x+y}{1-xy} & \text{if } x<0, y<0, xy>1\end{cases}$$

(ii) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\dfrac{x-y}{1+xy} & \text{if } xy>-1\\ \pi+\tan^{-1}\dfrac{x-y}{1+xy} & \text{if } x>0, y<0, xy<-1\\ -\pi+\tan^{-1}\dfrac{x-y}{1+xy} & \text{if } x<0, y>0, xy<-1\end{cases}$$

(iii) $$\tan^{-1}x_1+\tan^{-1}x_2+\tan^{-1}x_3+\cdots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5\cdots}{1-S_2+S_4\cdots}\right)$$

where S_k = sum of products of $x_1, x_2, \ldots, x_n$ taken k at a time

i.e. $S_1 = x_1+x_2+\cdots+x_n = \Sigma x_i$
$S_2 = x_1x_2+x_1x_3+\cdots = \Sigma x_ix_j$ etc.

8. (i) $$2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}) & \text{if } |x|\le\frac{1}{\sqrt2}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}) & \text{if } \frac{1}{\sqrt2}\le x\le 1\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}) & \text{if } -1\le x\le -\frac{1}{\sqrt2}\end{cases}$$

(ii) $$2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1) & \text{if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1) & \text{if } -1\le x\le 0\end{cases}$$

(iii) $$2\tan^{-1}x=\begin{cases}\tan^{-1}\dfrac{2x}{1-x^2} & \text{if } |x|<1\\ \pi+\tan^{-1}\dfrac{2x}{1-x^2} & \text{if } x>1\\ -\pi+\tan^{-1}\dfrac{2x}{1-x^2} & \text{if } x<-1\end{cases}$$

9. If $|x|\le 1$, then $2\tan^{-1}x=\sin^{-1}\dfrac{2x}{1+x^2}=\cos^{-1}\dfrac{1-x^2}{1+x^2}$

Note. If $|x|>1$, then change x to $\frac{1}{x}$ in the above.

10. (i) $\sin^{-1}x=\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, $|x|<1$

(ii) $\cos^{-1}x=\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$, $0<x\le 1$

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11. (i) $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3) & \text{if } |x| \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3) & \text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3) & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$$

(ii) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x) & \text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x) & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x) & \text{if } -1 \le x \le -\frac{1}{2} \end{cases}$$

(iii) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{if } |x| < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$$

12. (i) $\sin^{-1}x + \sin^{-1}y = \left(\sin^{-1} x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, $|x| \le 1, |y| \le 1$ and $x^2+y^2 \le 1$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, $|x| \le 1$, $|y| \le 1$ and $x^2+y^2 \le 1$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, $|x| \le 1$, $|y| \le 1$ and $x+y \ge 0$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ $|x| \le 1$, $|y| \le 1$ and $x < y$

1. The value of $\sin^{-1}(\cos x)$, $0 \le x \le \pi$, is

(a) $\pi - x$ (b) $x - \frac{\pi}{2}$ ✓(c) $\frac{\pi}{2} - x$ (d) $\pi - x$

2. The range of the function $\sin(\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$, is

(a) $[-1, 1]$ (b) $(-1, 1)$ (c) $\{0\}$ ✓(d) $\{1\}$

3. $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid for

(a) $-\pi \le x \le 0$ ✓(b) $0 \le x \le \pi$ (c) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (d) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

4. $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ is valid for

(a) $-\pi \le x \le 0$ (b) $0 \le x \le \pi$ ✓(c) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (d) none of these

5. $\sin\left(\cot^{-1}\left(\cot \frac{17\pi}{3}\right)\right)$ is equal to

✓(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

6. The principal value of $\sin^{-1}\left(\cos \frac{33\pi}{5}\right)$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{5}$ (c) $\frac{\pi}{10}$ ✓(d) $-\frac{\pi}{10}$

7. The principal value $\cos^{-1}\left(\sin \frac{10\pi}{7}\right)$ is

(a) $\frac{3\pi}{7}$ (b) $\frac{\pi}{14}$ ✓(c) $\frac{13\pi}{14}$ (d) none of these

8. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$ is equal to

(a) $\frac{2\pi}{3}$ ✓(b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π UD

9. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then the value of x is

✓(a) $\frac{1}{5}$ (b) 1 (c) 0 (d) $\frac{4}{5}$

10. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then the value of x is

(a) $-\frac{1}{2}$ ✓(b) $\frac{1}{2}$ (c) $\pm\frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}$

11. If $a \le \tan^{-1}x + \cot^{-1}x + \sin^{-1}x \le b$, then

(a) $a = \frac{\pi}{4}$ ✓(b) $a = 0$ (c) $b = \frac{\pi}{2}$ ✓(d) $b = \pi$ UD

12. If $h \le \sin^{-1}x + \cos^{-1}x + \tan^{-1}x \le k$, then

✓(a) $h = 0, k = \pi$ (b) $h = 0, k = \frac{\pi}{2}$

(c) $h = \frac{\pi}{2}, k = \pi$ (d) none of these

13. The value of $\tan^{-1}(\tan(-6))$ is UD

(a) -6 (b) $\pi-6$ (c) $2\pi-6$ (d) none of these

14. The value of $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 12)$ is UD

(a) 0 (b) π (c) $8\pi-24$ (d) none of these

15. The complete set of solutions of $\sin^{-1}(\sin 5) > x^2-4x$ is UD

(a) $|x-2| < \sqrt{9-2\pi}$ (b) $|x-2| > \sqrt{9-2\pi}$

(c) $|x| < \sqrt{9-2\pi}$ (d) $|x| > \sqrt{9-2\pi}$

16. If $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$, then the value of x is UD

(a) 4 (b) 5 (c) 1 (d) 3

17. If $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$, then the value of x is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) none of these

18. If $\sin^{-1}6x + \sin^{-1}6\sqrt{3}x = -\frac{\pi}{2}$, then the value of x is

(a) $\frac{1}{12}$ (b) $-\frac{1}{12}$ (c) $\pm\frac{1}{12}$ (d) none of these

19. The value of $\tan^{-1}5 + \tan^{-1}3 - \cot^{-1}\frac{4}{7}$ is

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) 0 (d) π

20. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is UD

(a) $\frac{6}{17}$ (b) $\frac{3}{17}$ (c) $\frac{4}{17}$ (d) $\frac{5}{17}$

21. The value of $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$ is

(a) $\frac{7}{17}$ (b) $-\frac{7}{17}$ (c) $\frac{14}{17}$ (d) $-\frac{14}{17}$

22. The value of $\sin^{-1}\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$ is

(a) $\frac{15}{14}$ (b) $-\frac{14}{15}$ (c) $\frac{14}{15}$ (d) none of these

23. The value of $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$ is

(a) $\frac{4+\sqrt{7}}{3}$ (b) $\frac{4-\sqrt{7}}{3}$ (c) $\frac{4\pm\sqrt{7}}{3}$ (d) none of these

24. If $\sin^{-1}x = 2\sin^{-1}\alpha$ has a solution, then UD

(a) $|\alpha| \geq \frac{1}{\sqrt{2}}$ (b) $|\alpha| \leq \frac{1}{\sqrt{2}}$ (c) all real values of α (d) $|\alpha| < \frac{1}{\sqrt{2}}$

25. If x satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for UD

(a) $\sin^{-1}x$ (b) $\sec^{-1}x$ (b) $\cos^{-1}x$ (d) none of these

26. If $3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{2}$, then the value of x is

(a) 1 (b) 3 (c) $\sqrt{3}$ (d) -3

27. If a ΔABC, if $A = \tan^{-1}2$ and $B = \tan^{-1}3$, then C is equal to UD

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) none of these

28. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then UD

(a) $x+y+z = 3xyz$ (b) $x+y+z = 2xyz$

(c) $x+y+z = xyz$ (d) none of these

29. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, then UD

(a) $x+y+z = 3xyz$ (b) $x+y+z = 2xyz$

(c) $xy+yz+zx = 1$ (d) none of these

30. If $\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \frac{\pi}{2}$, then $x+y+z$ is equal to

(a) xyz (b) $2xyz$ (c) $xy+yz+zx$ (d) none of these

31. If x, y, z are positive numbers, then

$$\tan^{-1}\sqrt{\frac{x(x+y+z)}{yz}} + \tan^{-1}\sqrt{\frac{y(x+y+z)}{zx}} + \tan^{-1}\sqrt{\frac{z(x+y+z)}{xy}}$$ UD

is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) none of these

32. An integral solution of the equation $\tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}3$ is

(a) $(2, 7)$ (b) $(4, -13)$ (c) $(5, -8)$ (d) $(1, 2)$ UD

33. The number of positive integral solutions of the equation

$$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$$ is UD

(a) 1 (b) 2 (c) 3 (d) none of these

34. If $0 < x < 1$, then $\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1\right]^{1/2}$ is equal to

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x ✓(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$ UD

35. If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the value of $\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right)$ is UD

(a) $\frac{x}{2}$ (b) $2x$ (c) $3x$ ✓(d) x

36. If $x = \cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha})$, then $\sin x =$ UD

✓(a) $\tan^2\frac{\alpha}{2}$ (b) $\cot^2\frac{\alpha}{2}$ (c) $\tan\alpha$ (d) $\cot\frac{\alpha}{2}$

37. The value of $\sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right)$ is UD

✓(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) none of these

38. The value of $\tan\left(\cos^{-1}\left(-\frac{2}{7}\right)-\frac{\pi}{2}\right)$ is equal to

✓(a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

39. The value of

$$\tan^{-1}\frac{c_1x-y}{c_1y+x}+\tan^{-1}\frac{c_2-c_1}{1+c_1c_2}+\tan^{-1}\frac{c_3-c_2}{1+c_2c_3}+\cdots+\tan^{-1}\frac{1}{c_n} \text{ is}$$

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✓(a) $\tan^{-1}\frac{x}{y}$ (b) $\tan^{-1}\frac{y}{x}$ (c) $\tan^{-1}x-\tan^{-1}y$ (d) none of these

40. If $a_1, a_2, a_3, \dots, a_n$ is an AP with common difference d, then the value of

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right] \text{ is}$$

(a) $\frac{(n-1)d}{a_1+a_n}$ ✓(b) $\frac{(n-1)d}{1+a_1a_n}$ (c) $\frac{nd}{1+a_1a_n}$ (d) $\frac{a_n-a_1}{a_n+a_1}$

* 41. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z = 3\pi$, then $xy+yz+zx$ is equal to

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(a) -3 (b) 0 ✓(c) 3 (d) π UD

42. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2-4xy\cos\alpha+y^2$ is equal to

(a) $a\sin 2\alpha$ (b) 4 ✓(c) $4\sin^2\alpha$ (d) $-4\sin^2\alpha$ UD

43. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \cdots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \cdots\right) = \frac{\pi}{2}$, $0 < |x| < \sqrt{2}$, then x is equal to UD

(a) $\frac{1}{2}$ ✓(b) 1 (c) $-\frac{1}{2}$ (d) -1

44. If $a\sin^{-1}x - b\cos^{-1}x = c$, then $a\sin^{-1}x + b\cos^{-1}x$ is equal to

(a) 0 (b) $\frac{\pi ab + c(b-a)}{a+b}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi ab + c(a-b)}{a+b}$ UD

45. If $A = 2\tan^{-1}(2\sqrt{2}-1)$ and $B = 3\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{3}{5}$, then

(a) $A = B$ (b) $A < B$ (c) $A > B$ (d) none of these UD

46. Match the statements in column I with statements in column II

Column I	Column II
(a) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then $xy + yz + zx$ is	(p) $2n$
(b) $\sum_{i=1}^{10} \cos^{-1}x_i = 0$, then $\sum_{i=1}^{10} x_i$ is	(q) $\sin^{-1}x - \frac{\pi}{6}$
(c) $\sum_{i=1}^{2n} \sin^{-1}x_i = n\pi$, then $\sum_{i=1}^{2n} x_i$ is	(r) 3
(d) $f(x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}\sqrt{1-x^2}\right)$, $-\frac{1}{2} \le x \le 1$ is	(s) 10

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47. Let (x, y) be point such that $\sin^{-1}ax + \cos^{-1}y + \cos^{-1}bxy = \frac{\pi}{2}$.
Match the statements in column I with statements in column II

column I	column II
(a) If $a = 1$ and $b = 0$, then (x, y)	(p) Lies on the circle $x^2 + y^2 = 1$
(b) If $a = 1$, and $b = 1$, then (x, y)	(q) Lies on $(x^2-1)(y^2-1) = 0$
(c) If $a = 1$ and $b = 2$, then (x, y)	(r) Lies on $y = x$
(d) If $a = 2$ and $b = 2$, then (x, y)	(s) Lies on $(4x^2-1)(y^2-1) = 0$

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48. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then x equals UD

✓(a) -1 (b) 1 (c) 0 (d) none of these

49. The minimum value of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ is equal to

(a) $\frac{\pi^3}{32}$ (b) $\frac{5\pi^3}{32}$ (c) $\frac{9\pi^3}{32}$ (d) $\frac{11\pi^3}{32}$ UD

50. The greatest value of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi^3}{8}$ ✓(c) $\frac{7\pi^3}{8}$ (d) none of these

51. The minimum value of $(\tan^{-1}x)^2 + (\cot^{-1}x)^2$ is

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(a) $\frac{\pi^2}{4}$ ✓(b) $\frac{\pi^2}{8}$ (c) $\frac{\pi^2}{16}$ (d) none of these

52. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then $x^{100} + y^{100} + z^{100} - 3$ is equal to

✓(a) 0 (b) -3 (c) 3 (d) none of these

53. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then the value of

$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}}$ is

✓(a) 0 (b) 1 (c) 2 (d) 3

54. It is given that $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$, $x > 0$ and $B = (\cos^{-1}t)^2 + (\sin^{-1}t)^2$, $t \in \left[0, \frac{1}{\sqrt{2}}\right]$ and $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $-1 \le x \le 1$ and $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in R$.
Answer the following questions:

(i) The interval in which A lies is

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(a) $\left[\frac{\pi^3}{7}, \frac{\pi^3}{2}\right)$ (b) $\left(\frac{\pi^3}{40}, \frac{\pi^3}{10}\right)$ ✓(c) $\left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right)$ (d) none of these

(ii) The maximum value of B is

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{16}$ ✓(c) $\frac{\pi^2}{4}$ (d) none of these

(iii) If the least value of A is m and the maximum value of B is M, then $\cot^{-1}\left(\cot\left(\frac{m - \pi M}{M}\right)\right) =$

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(a) $-\frac{7\pi}{8}$ (b) $\frac{7\pi}{8}$ (c) $-\frac{\pi}{8}$ ✓(d) $\frac{\pi}{8}$

55. The sum to infinity of the series

$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \cdots$ is

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(a) $\frac{\pi}{6}$ ✓(b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) none of these

56. Sum to infinity of the series

$\tan^{-1}\frac{1}{2\cdot1^2}+\tan^{-1}\frac{1}{2\cdot2^2}+\tan^{-1}\frac{1}{2\cdot3^2}+\dots$ is UD

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) none of these

57. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\dots+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}+\dots$ to ∞ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

58. $\cot^{-1}(2^2+\frac{1}{2})+\cot(2^3+\frac{1}{2^2})+\dots$ to ∞ is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}2$ (c) $\cot^{-1}2$ (d) none of these

59. The value of $\sum_{r=1}^{n}\tan^{-1}\frac{2r}{2+r^2+r^4}$ is

(a) $\tan^{-1}(n^2+n+1)$ (b) $\tan^{-1}(n^2+n+1)-\frac{\pi}{4}$

(c) $\tan^{-1}(n^2-n+1)-\frac{\pi}{4}$ (d) none of these

60. The value of $\sum_{r=1}^{n}\tan^{-1}\frac{1}{r^2-r+1}$ is

(a) 0 (b) $\tan^{-1}\frac{1}{n}$ (c) $\tan^{-1}n$ (d) none of these

61. The value of

$\operatorname{cosec}^{-1}\sqrt{5}+\operatorname{cosec}^{-1}\sqrt{65}+\operatorname{cosec}^{-1}\sqrt{325}+\dots$ to ∞ is

(a) π (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

62. The sum to n terms of the series UD

$\operatorname{cosec}^{-1}\sqrt{10}+\operatorname{cosec}^{-1}\sqrt{50}+\operatorname{cosec}^{-1}\sqrt{170}+\dots+\operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$ is

(a) 0 (b) ∞ (c) $\tan^{-1}(n+1)-\frac{\pi}{4}$ (d) $\cot^{-1}(n+1)-\frac{\pi}{4}$

63. $\sum_{r=1}^{n}\sin^{-1}\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}$ is UD

(a) $\tan^{-1}\sqrt{n}-\frac{\pi}{4}$ (b) $\tan^{-1}\sqrt{n+1}-\frac{\pi}{4}$ (c) $\tan^{-1}\sqrt{n}$ (d) $\tan^{-1}\sqrt{n+1}$

64. If $\cot^{-1}(\frac{n}{\pi})>\frac{\pi}{6}$, $n\in N$, then the maximum value of n can be UD

(a) 4 (b) 5 (c) 6 (d) none of these

65. Read the passage and answer the following questions.

If $\tan^{-1} x : \tan^{-1} y = 1 : 4$ (when $|x| < \tan \frac{\pi}{6}$), then

(i) the value of y as an algebraic function of x will be UD

(a) $\frac{4x(1+x^2)}{x^4 - 6x^2 + 1}$ ✓(b) $\frac{4x(1-x^2)}{x^4 - 6x^2+1}$ (c) $\frac{4x(1+x^2)}{x^4+6x^2+1}$ (d) none of these

(ii) The root of the equation $x^4 - 6x^2+1=0$ is UD

(a) $\tan \frac{\pi}{12}$ (b) $\tan \frac{\pi}{4}$ ✓(c) $\tan \frac{\pi}{8}$ (d) $\tan \frac{\pi}{16}$

66. $\cot^{-1}(1+\frac{3}{4}) + \cot^{-1}(2^2+\frac{3}{4}) + \cot^{-1}(3^2+\frac{3}{4}) + \dots$ to ∞ is

(a) $\frac{\pi}{4}$ (b) $\cot^{-1} 2$ ✓(c) $\tan^{-1} 2$ (d) none of these

67. If $2\tan^{-1} x + \sin^{-1} \frac{2x}{1+x^2}$ is independent of x, then UD

(a) $x \in [1, \infty)$ (b) $x \in [-1, 1]$ (c) $x \in (-\infty, -1]$ (d) none of these

68. Given that $\tan^{-1} \frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x, & |x| < 1 \\ -\pi + 2\tan^{-1} x, & x > 1 \\ \pi + 2\tan^{-1} x, & x < 1 \end{cases}$

$$\sin^{-1} \frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1} x, & x \le 1 \\ \pi - 2\tan^{-1} x, & x > 1 \\ -(\pi + 2\tan^{-1} x, & x < -1 \end{cases}$$

and $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for all $-1 \le x \le 1$.

Answer the following questions:

(i) $\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right)$ is independent of x when UD

(a) $x \in [1, \infty)$ (b) $x \in [-1, 1]$ ✓(c) $x \in [-2, 2]$ (d) $x \in (-3, 4)$

(ii) If $(x-1)(x^2+1) > 0$, then $\sin\left(\frac{1}{2}\tan^{-1} \frac{2x}{1-x^2} - \tan^{-1} x\right) =$

✓(a) -1 (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) none of these UD

(iii) If $\cos^{-1}\left(\frac{6x}{1+9x^2}\right) = -\frac{\pi}{2} + 2\tan^{-1} 3x$, then x belongs to UD

(a) $(-\infty, -1)$ (b) $(-\frac{1}{3}, \frac{1}{3})$ ✓(c) $(\frac{1}{3}, \infty)$ (d) none of these

69. The number of solutions of the equation

$$\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right) \text{ is}$$ UD

✓(a) 1 (b) 2 (c) 3 (d) 4

70. The range of the function $f(x) = \cos^{-1}(-\{x\})$, where $\{\ \}$ is fractional part function, is UD

(a) $\left(\frac{\pi}{2}, \pi\right)$ (b) $\left(\frac{\pi}{2}, \pi\right]$ ✓(c) $\left[\frac{\pi}{2}, \pi\right)$ (d) $\left(0, \frac{\pi}{2}\right]$

71. Match the statement of column I with values of column II

Column I	Column II
(a) The absolute difference of greatest and least value of $\sqrt{2}(\sin 2x - \cos 2x)$ is	(p) $\frac{\pi}{4}$
(b) The difference of greatest and least value of $x^2 - 4x + 3$, $x \in [1, 3]$ is	(q) $\frac{\pi}{6}$
(c) Greatest value of $\tan^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0, 1]$ is	(r) 4
(d) Absolute difference of greatest and least value of $\cos^{-1} x^2$, $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{2}\right]$ is	(s) 1

72. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Statement I : $f'(2) = \frac{2}{5}$

Statement II : $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x, \ \forall x > 1$

✓(a) Statement I is true, statement II is true ; statement II is a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is <u>Not</u> a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true.

1. $\sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \frac{\pi}{2} - x \quad (\because 0 \le x \le \pi)$

2. $\sin(\sin^{-1}x + \cos^{-1}x) = \sin\frac{\pi}{2} = 1.$

3. $\sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid if $0 \le x \le \pi$.

4. $\cos^{-1}(\sin x) = \frac{\pi}{2} - \sin^{-1}(\sin x) = \frac{\pi}{2} - x$ is valid if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

5. $\sin(\cot^{-1}(\cot(4\pi + \frac{5\pi}{3}))) = \sin(\cot^{-1}(\cot\frac{5\pi}{3})) = \sin(\cot^{-1}(\cot(\pi + \frac{2\pi}{3})))$

$= \sin(\cot^{-1}(\cot\frac{2\pi}{3})) = \sin\frac{2\pi}{3} = \sin(\pi - \frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

6. $\sin^{-1}(\cos\frac{33\pi}{5}) = \sin^{-1}(\cos(6\pi + \frac{3\pi}{5})) = \sin^{-1}(\cos\frac{3\pi}{5})$

$= \sin^{-1}(\cos(\frac{\pi}{2} + \frac{\pi}{10})) = \sin^{-1}(-\sin\frac{\pi}{10})$

$= \sin^{-1}(\sin(-\frac{\pi}{10})) = -\frac{\pi}{10}$

7. $\cos^{-1}(\sin\frac{10\pi}{7}) = \cos^{-1}(\sin(\frac{\pi}{2} + \frac{13\pi}{14})) = \cos^{-1}(\cos\frac{13\pi}{14}) = \frac{13\pi}{14}$

8. $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3} \Rightarrow (\frac{\pi}{2} - \cos^{-1}x) + (\frac{\pi}{2} - \cos^{-1}y) = \frac{2\pi}{3}$

$\Rightarrow \pi - \frac{2\pi}{3} = \cos^{-1}x + \cos^{-1}y \Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$

9. $\sin(\sin^{-1}\frac{1}{5} + \cos^{-1}x) = 1 \Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$

$\Rightarrow \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x \Rightarrow \sin^{-1}\frac{1}{5} = \sin^{-1}x \Rightarrow x = \frac{1}{5}.$

10. $4\sin^{-1}x + \cos^{-1}x = \pi \Rightarrow 3\sin^{-1}x + (\sin^{-1}x + \cos^{-1}x) = \pi$

$\Rightarrow 3\sin^{-1}x + \frac{\pi}{2} = \pi \Rightarrow 3\sin^{-1}x = \frac{\pi}{2} \Rightarrow \sin^{-1}x = \frac{\pi}{6}$

$\Rightarrow x = \sin\frac{\pi}{6} \Rightarrow x = \frac{1}{2}$

11. $\tan^{-1}x + \cot^{-1}x + \sin^{-1}x = \frac{\pi}{2} + \sin^{-1}x$:

We know that $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2} \Rightarrow 0 \le \frac{\pi}{2} + \sin^{-1}x \le \pi$

$\Rightarrow a = 0$ and $b = \pi$.

12. $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x = \frac{\pi}{2} + \tan^{-1}x.$

We know that $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} + \tan^{-1}x < \pi$

$\Rightarrow h = 0$ and $k = \pi$

13. $\tan(2\pi - 6) = -\tan 6 = \tan(-6)$; $2\pi - 6 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = 2\pi - 6$.

14. $\cos(4\pi-12) = \cos(-12) = \cos 12 \; ; \; 4\pi - 12 \in [0, \pi]$

$\therefore \cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi-12)) = 4\pi - 12$

$\sin(-4\pi+12) = \sin 12 \; ; \; -4\pi+12 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \sin^{-1}(\sin 12) = \sin^{-1}(\sin(-4\pi+12)) = -4\pi+12.$

$\therefore \cos^{-1}(\cos 12) - \sin^{-1}(\sin 12) = 4\pi - 12 - (-4\pi+12) = 8\pi - 24.$

15. $\sin(-2\pi+5) = \sin 5 \; ; \; -2\pi+5 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \sin^{-1}(\sin 5) = \sin^{-1}(\sin(-2\pi+5)) = -2\pi+5.$

$\therefore \sin^{-1}(\sin 5) > x^2-4x \Rightarrow -2\pi+5 > x^2-4x$

$\Rightarrow x^2-4x < 5-2\pi \Rightarrow x^2-4x+4 < 9-2\pi$

$\Rightarrow (x-2)^2 < 9-2\pi \Rightarrow |x-2| < \sqrt{9-2\pi} \quad (\because x^2 = |x|^2)$

16. $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{x}{5} = \frac{\pi}{2} - \sin^{-1}\frac{4}{5}$ | $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$

$\Rightarrow \frac{x}{5} = \sin\left(\frac{\pi}{2} - \sin^{-1}\frac{4}{5}\right) \Rightarrow \frac{x}{5} = \cos\left(\sin^{-1}\frac{4}{5}\right)$

$\Rightarrow \frac{x}{5} = \sqrt{1-\left(\frac{4}{5}\right)^2}$ | $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

$\Rightarrow \frac{x}{5} = \frac{3}{5} \Rightarrow x = 3.$

17. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x) \Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x \Rightarrow \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} = \frac{1}{\sin x} \Rightarrow \cot x = 1$

$\Rightarrow x = \frac{\pi}{4}.$

18. $\sin^{-1}6x + \sin^{-1}6\sqrt{3}x = -\frac{\pi}{2} \Rightarrow \sin^{-1}6\sqrt{3}x = -\frac{\pi}{2} - \sin^{-1}6x$

$\Rightarrow 6\sqrt{3}x = \sin\left(-\frac{\pi}{2} - \sin^{-1}6x\right) = -\sin\left(\frac{\pi}{2} + \sin^{-1}6x\right) = -\cos(\sin^{-1}6x)$

$\Rightarrow 6\sqrt{3}x = -\sqrt{1-(6x)^2} \Rightarrow 6\sqrt{3}x = -\sqrt{1-36x^2}$

$\Rightarrow 108x^2 = 1-36x^2 \Rightarrow 144x^2 = 1 \Rightarrow x = \pm\frac{1}{12}.$

But $x = \frac{1}{12}$ does not satisfy the given equation.

$\therefore x = -\frac{1}{12}$

19. $\tan^{-1}5 + \tan^{-1}3 - \cot^{-1}\frac{4}{7} = \pi + \tan^{-1}\frac{5+3}{1-5\times 3} - \cot^{-1}\frac{4}{7} \quad (xy>1)$

$= \pi + \tan^{-1}\left(\frac{8}{-14}\right) - \cot^{-1}\frac{4}{7} = \pi - \tan^{-1}\frac{4}{7} - \cot^{-1}\frac{4}{7}$

$= \pi - \left(\tan^{-1}\frac{4}{7} + \cot^{-1}\frac{4}{7}\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

20. $\operatorname{cosec}^{-1}\frac{5}{3} = \sin^{-1}\frac{3}{5} = \tan^{-1}\dfrac{\frac{3}{5}}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \tan^{-1}\frac{3}{4}$.

$\therefore \operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3} = \tan^{-1}\dfrac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}} \quad (xy<1)$

$= \tan^{-1}\frac{17}{6}$.

$\therefore \cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right) = \cot\left(\tan^{-1}\frac{17}{6}\right) = \cot\left(\cot^{-1}\frac{6}{17}\right) = \frac{6}{17}$.

21. $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \tan\left(\tan^{-1}\left(\dfrac{2\cdot\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right) = \tan\left(\tan^{-1}\frac{5}{12} - \tan^{-1}1\right)$

$= \tan\left(\tan^{-1}\left(\dfrac{\frac{5}{12}-1}{1+\frac{5}{12}\cdot 1}\right)\right) = \tan\left(\tan^{-1}\left(-\frac{7}{17}\right)\right) = -\frac{7}{17}$

22. Let $\tan^{-1}\frac{1}{3} = \alpha \Rightarrow \tan\alpha = \frac{1}{3}$; $\tan^{-1}2\sqrt{2} = \beta \Rightarrow \tan\beta = 2\sqrt{2}$

$\therefore \sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right) = \sin 2\alpha + \cos\beta = \dfrac{2\tan\alpha}{1+\tan^2\alpha} + \dfrac{1}{\sqrt{1+\tan^2\beta}}$

$= \dfrac{2\cdot\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2} + \dfrac{1}{\sqrt{1+(2\sqrt{2})^2}} = \frac{2}{3}\cdot\frac{9}{10} + \frac{1}{3} = \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$.

23. Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \alpha \Rightarrow \sin 2\alpha = \frac{3}{4}$

(As $0 < \sin^{-1}\frac{3}{4} < \frac{\pi}{2}$, $0 < \frac{1}{2}\sin^{-1}\frac{3}{4} < \frac{\pi}{4}$
$\Rightarrow 0 < \alpha < \frac{\pi}{4} \Rightarrow 0 < \tan\alpha < 1$)

$\Rightarrow \dfrac{2\tan\alpha}{1+\tan^2\alpha} = \frac{3}{4} \Rightarrow 3\tan^2\alpha - 8\tan\alpha + 3 = 0$

$\Rightarrow \tan\alpha = \dfrac{8 \pm \sqrt{64-4.3.3}}{2\cdot 3} = \dfrac{8\pm 2\sqrt{7}}{6} = \dfrac{4\pm\sqrt{7}}{3}$.

But $0 < \tan\alpha < 1 \Rightarrow \tan\alpha = \dfrac{4-\sqrt{7}}{3}$.

24. We know that $2\sin^{-1}\alpha = \sin^{-1}\left(2\alpha\sqrt{1-\alpha^2}\right)$ for $|\alpha| \le \frac{1}{\sqrt{2}}$

So $\sin^{-1}x = \sin^{-1}2\alpha\sqrt{1-\alpha^2} \Rightarrow x = 2\alpha\sqrt{1-\alpha^2}$
$\Rightarrow$ the given equation has solution $x = 2\alpha\sqrt{1-\alpha^2}$ for $|\alpha| \le \frac{1}{\sqrt{2}}$

25. $x^2 - x - 2 > 0 \Rightarrow (x+1)(x-2) > 0$

$\Rightarrow x < -1$ or $x > 2$

We know that domain of $\sec^{-1}x$ is $(-\infty, -1] \cup [1, \infty)$, so $\sec^{-1}x$ will have solutions for $x < -1$ or $x > 2$.

26. $\tan^{-1}\frac{1}{2+\sqrt{3}} = \tan^{-1}\frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \tan^{-1}(2-\sqrt{3}) = \tan^{-1}(\tan\frac{\pi}{12}) = \frac{\pi}{12}$

$\therefore 3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{2} \Rightarrow 3\times\frac{\pi}{12} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{2}$

$\Rightarrow \frac{\pi}{4} - \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{x} \Rightarrow \tan^{-1}1 - \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{x}$

$\Rightarrow \tan^{-1}\frac{1-\frac{1}{2}}{1+1.\frac{1}{2}} = \tan^{-1}\frac{1}{x} \Rightarrow \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{x}$

$\Rightarrow \frac{1}{3} = \frac{1}{x} \Rightarrow x = 3.$

27. $A+B = \tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\frac{2+3}{1-2.3} \quad (\because xy>1)$

$= \pi + \tan^{-1}(-1) = \pi + \tan^{-1}(\tan(-\frac{\pi}{4})) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

We know that in a ΔABC, $A+B+C = \pi$

$\Rightarrow \frac{3\pi}{4} + C = \pi \Rightarrow C = \frac{\pi}{4}.$

28. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi \Rightarrow \tan^{-1}x + \tan^{-1}y = \pi - \tan^{-1}z$

$\Rightarrow \tan^{-1}\frac{x+y}{1-xy} = \pi - \tan^{-1}z \Rightarrow \frac{x+y}{1-xy} = \tan(\pi - \tan^{-1}z)$

$\Rightarrow \frac{x+y}{1-xy} = -\tan(\tan^{-1}z) \Rightarrow \frac{x+y}{1-xy} = -z \Rightarrow x+y+z = xyz.$

29. $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2} - \tan^{-1}z \Rightarrow \tan^{-1}\frac{x+y}{1-xy} = \frac{\pi}{2} - \tan^{-1}z$

$\Rightarrow \frac{x+y}{1-xy} = \tan(\frac{\pi}{2} - \tan^{-1}z) = \cot(\tan^{-1}z) = \cot(\cot^{-1}\frac{1}{z})$

$\Rightarrow \frac{x+y}{1-xy} = \frac{1}{z} \Rightarrow xy+yz+zx = 1.$

30. $\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \frac{\pi}{2} \Rightarrow \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{y} = \frac{\pi}{2} - \cot^{-1}z$

$\Rightarrow \tan^{-1}\frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{x}\cdot\frac{1}{y}} = \frac{\pi}{2} - \cot^{-1}z \Rightarrow \frac{y+x}{xy-1} = \tan(\frac{\pi}{2} - \cot^{-1}z)$

$\Rightarrow \frac{x+y}{xy-1} = \cot(\cot^{-1}z) \Rightarrow \frac{x+y}{xy-1} = z$

$\Rightarrow x+y = xyz - z \Rightarrow x+y+z = xyz.$

31. Let $x+y+z = r$, then

given expression $= \tan^{-1}\sqrt{\frac{xr}{yz}} + \tan^{-1}\sqrt{\frac{yr}{zx}} + \tan^{-1}\sqrt{\frac{zr}{xy}}.$

We note that $\sqrt{\frac{xr}{yz}}\cdot\sqrt{\frac{yr}{zx}} = \frac{r}{z} = \frac{x+y+z}{z} = \frac{x+y}{z} + 1 > 1$

$\therefore \tan^{-1}\sqrt{\frac{xr}{yz}} + \tan^{-1}\sqrt{\frac{yr}{zx}} = \pi + \tan^{-1}\frac{\sqrt{\frac{xr}{yz}} + \sqrt{\frac{yr}{zx}}}{1 - \sqrt{\frac{xr}{yz}}\sqrt{\frac{yr}{zx}}} = \pi + \tan^{-1}\frac{\sqrt{\frac{r}{xyz}}(x+y)}{1-\frac{r}{z}}$

$$= \pi + \tan^{-1}\frac{\sqrt{\frac{\lambda z}{xy}}(x+y)}{z-(x+y+z)} = \pi + \tan^{-1}\left(-\sqrt{\frac{\lambda z}{xy}}\right) = \pi - \tan^{-1}\sqrt{\frac{\lambda z}{xy}}.$$

$\therefore$ Given expression $= \left(\pi - \tan^{-1}\sqrt{\frac{\lambda z}{xy}}\right) + \tan^{-1}\sqrt{\frac{\lambda z}{xy}} = \pi$.

32. $\tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}3 \Rightarrow \tan^{-1}\frac{1}{y} = \tan^{-1}3 - \tan^{-1}x$

$\Rightarrow \tan^{-1}\frac{1}{y} = \tan^{-1}\frac{3-x}{1+3x}$, $3x > -1$ i.e. $x > -\frac{1}{3}$

$\Rightarrow \frac{1}{y} = \frac{3-x}{1+3x} \Rightarrow y = \frac{3x+1}{3-x}$, $x > -\frac{1}{3}$ but $x \in \underline{\underline{I}}$

When $x=1$, $y=2$; when $x=2$, $y=7$;

when $x=4$, $y=-13$; when $x=5$, $y=-8$

$\therefore$ $(1,2)$, $(2,7)$, $(4,-13)$ and $(5,-8)$ are all solutions

33. $\cos^{-1}\frac{y}{\sqrt{1+y^2}} = \tan^{-1}\frac{\sqrt{1-\frac{y^2}{1+y^2}}}{\frac{y}{\sqrt{1+y^2}}} = \tan^{-1}\frac{1}{y}$ $\quad \left(\because \cos^{-1}x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}\right)$

and $\sin^{-1}\frac{3}{\sqrt{10}} = \tan^{-1}\frac{\frac{3}{\sqrt{10}}}{\sqrt{1-\left(\frac{3}{\sqrt{10}}\right)^2}} = \tan^{-1}3$. $\quad \left(\because \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}\right)$

$\therefore \tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}} \Rightarrow \tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}3$

$\Rightarrow y = \frac{3x+1}{3-x}$, $x > -\frac{1}{3}$ but $x \in \underline{N}$. (See Q. 32)

For x and y to be both positive, the only possible values of x are 1 and 2.

When $x=2$, $y=7$; when $x=1$, $y=2$.

$\therefore$ the equation has two positive integral solutions $(2,7)$ and $(1,2)$

34. Note that, $\cot^{-1}x = \cos^{-1}\frac{x}{\sqrt{1+x^2}}$ and $\cot^{-1}x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

(Let $\cot^{-1}x = \alpha \Rightarrow \cot\alpha = x$ $= \operatorname{cosec}\alpha = \sqrt{1+x^2}$)

$x\cos(\cot^{-1}x) + \sin(\cot^{-1}x) = x\cos\left(\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right) + \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)$

$$= x\cdot\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} = \frac{x^2+1}{\sqrt{1+x^2}} = \sqrt{x^2+1}$$

$\therefore \sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1\right]^{1/2}$

$$= \sqrt{1+x^2}\left\{(\sqrt{x^2+1})^2 - 1\right\}^{1/2} = \sqrt{1+x^2}\,(x^2+1-1)^{1/2}$$

$$= x\sqrt{1+x^2}$$

35. Given expression $= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\dfrac{3 \cdot \frac{2\tan x}{1+\tan^2 x}}{5 + 3 \cdot \frac{1-\tan^2 x}{1+\tan^2 x}}\right)$

$$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\tan x}{4+\tan^2 x}\right)$$

$$= \tan^{-1}\left(\dfrac{\frac{\tan x}{4} + \frac{3\tan x}{4+\tan^2 x}}{1 - \frac{\tan x}{4} \cdot \frac{3\tan x}{4+\tan^2 x}}\right) \quad \Big| \because \frac{\tan x}{4} \cdot \frac{3\tan x}{4+\tan^2 x} = \frac{3}{4} \cdot \frac{\tan^2 x}{4+\tan^2 x} < 1$$

$$= \tan^{-1}\left(\frac{16\tan x + \tan^3 x}{16+\tan^2 x}\right) = \tan^{-1}(\tan x) = x$$

36. $x = \left(\frac{\pi}{2} - \tan^{-1}(\sqrt{\cos\alpha})\right) - \tan^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2} - 2\tan^{-1}(\sqrt{\cos\alpha})$

$$= \frac{\pi}{2} - \cos^{-1} \frac{1-(\sqrt{\cos\alpha})^2}{1+(\sqrt{\cos\alpha})^2} = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)$$

$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{2\sin^2\alpha/2}{2\cos^2\alpha/2}\right) = \frac{\pi}{2} - \cos^{-1}\left(\tan^2\frac{\alpha}{2}\right)$$

$$\Rightarrow \cos^{-1}\left(\tan^2\frac{\alpha}{2}\right) = \frac{\pi}{2} - x \Rightarrow \tan^2\frac{\alpha}{2} = \cos\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \tan^2\frac{\alpha}{2} = \sin x.$$

37. $\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} = \sin^{-1}\sqrt{\frac{4-2\sqrt{3}}{8}} = \sin^{-1}\sqrt{\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2} = \sin^{-1}\frac{\sqrt{3}-1}{2\sqrt{2}}$

$$= \sin^{-1}\left(\sin\frac{\pi}{12}\right) = \frac{\pi}{12};$$

$\cos^{-1}\frac{\sqrt{12}}{4} = \cos^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$; $\sec^{-1}\sqrt{2} = \sec^{-1}\left(\sec\frac{\pi}{4}\right) = \frac{\pi}{4}$.

$\therefore$ Given expression $= \sin^{-1}\left(\cot\left(\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right)\right) = \sin^{-1}\left(\cot\frac{\pi}{2}\right)$

$$= \cancel{\sin^{-1}\sin} = \sin^{-1} 0 = \sin^{-1}(\sin 0) = 0.$$

38. $\tan\left(\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right) = \tan\left(\left(\pi - \cos^{-1}\frac{2}{7}\right) - \frac{\pi}{2}\right)$

$$= \tan\left(\frac{\pi}{2} - \cos^{-1}\frac{2}{7}\right) = \tan\left(\sin^{-1}\frac{2}{7}\right) = \tan\left(\tan^{-1}\frac{2}{3\sqrt{5}}\right) = \frac{2}{3\sqrt{5}}$$

$$\left(\sin^{-1}\frac{2}{7} = \tan^{-1}\frac{\frac{2}{7}}{\sqrt{1-\left(\frac{2}{7}\right)^2}} = \tan^{-1}\frac{2}{\sqrt{45}} = \tan^{-1}\frac{2}{3\sqrt{5}}\right)$$

39. Given series $= \tan^{-1}\dfrac{\frac{x}{y} - \frac{1}{c_1}}{1 + \frac{x}{y}\cdot\frac{1}{c_1}} + \tan^{-1}\dfrac{\frac{1}{c_1} - \frac{1}{c_2}}{1 + \frac{1}{c_1}\cdot\frac{1}{c_2}} + \cdots + \tan^{-1}\dfrac{\frac{1}{c_{n-1}} - \frac{1}{c_n}}{1 + \frac{1}{c_{n-1}}\cdot\frac{1}{c_n}} + \tan^{-1}\frac{1}{c_n}$

$$= \left(\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1}\right) + \left(\tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2}\right) + \cdots + \left(\tan^{-1}\frac{1}{c_{n-1}} - \tan^{-1}\frac{1}{c_n}\right) + \tan^{-1}\frac{1}{c_n}$$

$$= \tan^{-1}\frac{x}{y}.$$

40. As $a_1, a_2, a_3, \ldots, a_n$ is an AP with common difference d,

$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = d.$

Now $\tan^{-1} \dfrac{d}{1+a_1a_2} = \tan^{-1} \dfrac{a_2-a_1}{1+a_1a_2} = \tan^{-1} a_2 - \tan^{-1} a_1$

$\tan^{-1} \dfrac{d}{1+a_2a_3} = \tan^{-1} \dfrac{a_3-a_2}{1+a_2a_3} = \tan^{-1} a_3 - \tan^{-1} a_2$

$\vdots \quad \vdots \quad \vdots$

$\tan^{-1} \dfrac{d}{1+a_{n-1}a_n} = \tan^{-1} \dfrac{a_n-a_{n-1}}{1+a_{n-1}a_n} = \tan^{-1} a_n - \tan^{-1} a_{n-1}$

Adding these, we get

$\tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) + \cdots + \tan^{-1}\left(\dfrac{d}{1+a_{n-1}a_n}\right) = \tan^{-1} a_n - \tan^{-1} a_1$

$= \tan^{-1} \dfrac{a_n-a_1}{1+a_1a_n}$ $\quad \big| \; a_n = a_1 + (n-1)d \Rightarrow a_n - a_1 = (n-1)d$

$= \tan^{-1} \dfrac{(n-1)d}{1+a_1a_n}.$

$\therefore$ Given expression $= \tan\left(\tan^{-1} \dfrac{(n-1)d}{1+a_1a_n}\right) = \dfrac{(n-1)d}{1+a_1a_n}.$

41. We know that $0 \le \cos^{-1} x \le \pi$, so max. value of $\cos^{-1} x = \pi$.

$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$

$\Rightarrow \cos^{-1} x = \pi, \quad \cos^{-1} y = \pi, \quad \cos^{-1} z = \pi$

$\Rightarrow x = \cos\pi, \quad y = \cos\pi, \quad z = \cos\pi \Rightarrow x = -1, y = -1, z = -1$

$\therefore xy + yz + zx = (-1)(-1) + (-1)(-1) + (-1)(-1) = 3.$

42. $\cos^{-1} x - \cos^{-1} \dfrac{y}{2} = \alpha \Rightarrow \cos^{-1} x = \alpha + \cos^{-1} \dfrac{y}{2}$

$\Rightarrow x = \cos\left(\alpha + \cos^{-1} \dfrac{y}{2}\right)$

$\Rightarrow x = \cos\alpha \cos\left(\cos^{-1} \dfrac{y}{2}\right) - \sin\alpha \sin\left(\cos^{-1} \dfrac{y}{2}\right)$

$\Rightarrow x = \cos\alpha \cdot \dfrac{y}{2} - \sin\alpha \cdot \sqrt{1 - \left(\dfrac{y}{2}\right)^2}$

$\Rightarrow \sin\alpha \sqrt{1 - \dfrac{y^2}{4}} = x - \dfrac{y}{2}\cos\alpha$

$\Rightarrow \sin^2\alpha \left(1 - \dfrac{y^2}{4}\right) = x^2 + \dfrac{y^2}{4}\cos^2\alpha - 2x \cdot \dfrac{y}{2}\cos\alpha$

$\Rightarrow 4\sin^2\alpha = 4x^2 + y^2(\cos^2\alpha + \sin^2\alpha) - 4xy\cos\alpha$

$\Rightarrow 4\sin^2\alpha = 4x^2 + y^2 - 4xy\cos\alpha.$

43. $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \cdots\right) = \frac{\pi}{2} - \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \cdots\right)$

$\Rightarrow \sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} \cdots\right) = \sin^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \cdots\right)$

$\Rightarrow x - \frac{x^2}{2} + \frac{x^3}{4} - \cdots = x^2 - \frac{x^4}{4} + \frac{x^6}{4}$

GP with ratio $-\frac{x^2}{2}$

(GP with ratio $-\frac{x}{2}$
as $0 < |x| < \sqrt{2}$, so $\left|-\frac{x}{2}\right| < 1$ and $\left|-\frac{x^2}{2}\right| < 1$)

$\Rightarrow \frac{x}{1-\left(-\frac{x}{2}\right)} = \frac{x^2}{1-\left(-\frac{x^2}{2}\right)}$ $\left(S_\infty = \frac{a}{1-r}, |r|<1\right)$

$\Rightarrow \frac{2x}{2+x} = \frac{2x^2}{2+x^2} \Rightarrow \frac{1}{2+x} = \frac{x}{2+x^2}$ $(\because x \neq 0)$

$\Rightarrow 2x + x^2 = 2 + x^2 \Rightarrow 2x = 2 \Rightarrow x = 1.$

44. $a\sin^{-1}x - b\cos^{-1}x = c$ ---(i) (given)

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ --(ii) (we know it)

Multiply (ii) by b and add to (i) —

$(a+b)\sin^{-1}x = c + b\frac{\pi}{2} \Rightarrow \sin^{-1}x = \frac{2c+b\pi}{2(a+b)}$ ---(iii)

Multiplying (ii) by a and subtracting (i) from it, we get

$(a+b)\cos^{-1}x = a\frac{\pi}{2} - c \Rightarrow \cos^{-1}x = \frac{a\pi - 2c}{2(a+b)}$ ---(iv)

From (iii) and (iv), we get

$a\sin^{-1}x + b\cos^{-1}x = \frac{a(2c+b\pi) + b(a\pi - 2c)}{2(a+b)} = \frac{ab\pi + c(a-b)}{a+b}.$

45. We know that $2\sqrt{2}-1 > \sqrt{3} \Rightarrow 2\sqrt{2}-1 > \tan\frac{\pi}{3}$

$\Rightarrow \tan^{-1}(2\sqrt{2}-1) > \tan^{-1}\left(\tan\frac{\pi}{3}\right)$ ($\because \tan^{-1}x$ is an increasing function)

$\Rightarrow \tan^{-1}(2\sqrt{2}-1) > \frac{\pi}{3} \Rightarrow 2\tan^{-1}(2\sqrt{2}-1) > \frac{2\pi}{3} \Rightarrow A > \frac{2\pi}{3}.$

$B = 3\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{3}{5} = \sin^{-1}\left(3\cdot\frac{1}{3} - 4\cdot\left(\frac{1}{3}\right)^3\right) + \sin^{-1}\frac{3}{5}$

$= \sin^{-1}\frac{23}{27} + \sin^{-1}\frac{3}{5} < \frac{\pi}{3} + \frac{\pi}{4} \Rightarrow B < \frac{7\pi}{12}$

($\because \frac{23}{27} = 0.85 < 0.86$ i.e. $\frac{23}{27} < \frac{\sqrt{3}}{2} \Rightarrow \frac{23}{27} < \sin\frac{\pi}{3}$

$\Rightarrow \sin^{-1}\left(\frac{23}{27}\right) < \sin^{-1}\left(\sin\frac{\pi}{3}\right)$ i.e. $\sin^{-1}\frac{23}{27} < \frac{\pi}{3}$ and $\frac{3}{5} < \frac{1}{\sqrt{2}} \Rightarrow \frac{3}{5} < \sin\frac{\pi}{4}$

$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) < \sin^{-1}\left(\sin\frac{\pi}{4}\right) \Rightarrow \sin^{-1}\frac{3}{5} < \frac{\pi}{4}$)

As $\frac{2\pi}{3} > \frac{7\pi}{12} \Rightarrow A > B.$

(c) See Q.41. $xy+yz+zx=3$, so (a) $\leftrightarrow$ (r)

(b) As $0 \le \cos^{-1}x_i \le \pi$, so minimum value of $\cos^{-1}x_i = 0$

$\sum_{i=1}^{10} \cos^{-1}x_i \Rightarrow \cos^{-1}x_i = 0$, for $i = 1, 2, 3, \cdots, 10$

$\Rightarrow x_i = \cos 0 \Rightarrow x_i = 1$

$\therefore \sum_{i=1}^{10} x_i = 1 + 1 + \cdots$ to 10 terms $= 10$

$\therefore$ (b) $\leftrightarrow$ (s)

(c) As $-\frac{\pi}{2} \le \sin^{-1}x_i \le \frac{\pi}{2}$, so maximum value of $\sin^{-1}x_i = \frac{\pi}{2}$

$\sum_{i=0}^{2n} \sin^{-1}x_i = n\pi \Rightarrow \sum_{i=1}^{2n} \sin^{-1}x_i = \frac{\pi}{2} + \frac{\pi}{2} + \cdots$ to $2n$ terms

$\Rightarrow \sin^{-1}x_i = \frac{\pi}{2}$, $i = 1, 2, 3, \cdots, 2n$

$\Rightarrow x_i = \sin\frac{\pi}{2} \Rightarrow x_i = 1$.

$\therefore \sum_{i=1}^{2n} x_i = 1 + 1 + 1 + \cdots$ to $2n$ terms $= 2n$

$\therefore$ (c) $\leftrightarrow$ (p)

(d) Let $x = \sin\alpha$, then $\sqrt{1-x^2} = \sqrt{1-\sin^2\alpha} = \cos\alpha$.

$f(x) = \sin^{-1}\left(\sin\alpha \cdot \cos\frac{\pi}{6} - \cos\alpha \cdot \sin\frac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(\alpha - \frac{\pi}{6}\right)\right)$

$= \alpha - \frac{\pi}{6} = \sin^{-1}x - \frac{\pi}{6}$.

$\therefore$ (d) $\leftrightarrow$ (q)

47. $\sin^{-1}ax + \cos^{-1}y + \cos^{-1}bxy = \frac{\pi}{2} \Rightarrow \cos^{-1}y + \cos^{-1}bxy = \frac{\pi}{2} - \sin^{-1}ax$

$\Rightarrow \cos^{-1}y + \cos^{-1}bxy = \cos^{-1}ax$.

(Let $\cos^{-1}y = \alpha$, $\cos^{-1}bxy = \beta$ and $\cos^{-1}ax = \gamma$

$\Rightarrow y = \cos\alpha$, $bxy = \cos\beta$ and $ax = \cos\gamma$)

$\Rightarrow \alpha + \beta = \gamma \Rightarrow \beta = \gamma - \alpha$

$\Rightarrow \cos\beta = \cos(\gamma - \alpha)$

$\Rightarrow \cos\beta = \cos\gamma\cos\alpha + \sin\gamma\sin\alpha$

$\Rightarrow bxy = axy + \sin\gamma\sin\alpha$

$\Rightarrow \sin\gamma\sin\alpha = bxy - axy$

$\Rightarrow \sin^2\gamma\sin^2\alpha = ((b-a)xy)^2$

$\Rightarrow (1-\cos^2\gamma)(1-\cos^2\alpha) = (b-a)^2x^2y^2$

$\Rightarrow (1-a^2x^2)(1-y^2) = (b-a)^2x^2y^2 \quad \cdots$ (i)

(a) Putting $a=1$ and $b=0$ in (i), we get

$(1-x^2)(1-y^2) = (0-1)^2x^2y^2$

$\Rightarrow (1-x^2)(1-y^2) = x^2y^2 \Rightarrow 1-x^2-y^2=0 \Rightarrow x^2+y^2=1$

$\Rightarrow$ point (x,y) lies on the circle $x^2+y^2=1$

$\therefore$ (a) $\leftrightarrow$ (p)

(b) Putting $a=1$ and $b=1$ in (i), we get

$(1-x^2)(1-y^2)=0 \Rightarrow (x^2-1)(y^2-1)=0$

$\Rightarrow$ point (x,y) lies on $(x^2-1)(y^2-1)=0$

$\therefore$ (b) $\leftrightarrow$ (q)

(c) Putting $a=1$ and $b=2$ in (i), we get

$(1-x^2)(1-y^2) = (2-1)^2x^2y^2$

$\Rightarrow (1-x^2)(1-y^2) = x^2y^2 \Rightarrow x^2+y^2=1$

$\Rightarrow$ point (x,y) lies on the circle $x^2+y^2=1$

$\therefore$ (c) $\leftrightarrow$ (p)

(d) Putting $a=2$ and $b=2$ in (i), we get

$(1-4x^2)(1-y^2) = (2-2)^2x^2y^2$

$\Rightarrow (4x^2-1)(y^2-1)=0 \Rightarrow$ point (x,y) lies on $(4x^2-1)(y^2-1)=0$

$\therefore$ (d) $\leftrightarrow$ (s)

48. $(\tan^{-1}x)^2+(\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1}x+\cot^{-1}x)^2 - 2\tan^{-1}x\cot^{-1}x = \frac{5\pi^2}{8}$

$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right) = \frac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$

$\Rightarrow 2y^2-\pi y-\frac{3\pi^2}{8}=0$ where $y=\tan^{-1}x$

$\Rightarrow 16y^2-8\pi y-3\pi^2=0 \Rightarrow (4y+\pi)(4y-3\pi)=0$

$\Rightarrow y = -\frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow \tan^{-1}x = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$

but $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$, so $\tan^{-1}x = -\frac{\pi}{4}$

$\Rightarrow x = \tan\left(-\frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1.$

49. $(\sin^{-1}x)^3+(\cos^{-1}x)^3 = (\sin^{-1}x+\cos^{-1}x)^3 - 3\sin^{-1}x\cos^{-1}x(\sin^{-1}x+\cos^{-1}x)$

$= \left(\frac{\pi}{2}\right)^3 - 3\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)\cdot\frac{\pi}{2}$

$= \frac{\pi^3}{8} - \frac{3\pi}{2}\left(\frac{\pi}{2}\sin^{-1}x-(\sin^{-1}x)^2\right)$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left((\sin^{-1}x)^2-\frac{\pi}{2}\sin^{-1}x\right)$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left(\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right)$$

$$= \frac{\pi^3}{8} - \frac{3\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$$

$\therefore$ Minimum value $= \dfrac{\pi^3}{32}$ $\quad \left(\because \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \geq 0\right)$

50. $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \dfrac{\pi^3}{32} + \dfrac{3\pi}{2}\left(\sin^{-1}x - \dfrac{\pi}{4}\right)^2$ (See Q.49).

Maximum value of $\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$ is attained at $\sin^{-1}x = -\frac{\pi}{2}$

$\therefore$ Maximum value of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \dfrac{\pi^3}{32} + \dfrac{3\pi}{2}\left(-\dfrac{\pi}{2} - \dfrac{\pi}{4}\right)^2$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\cdot\left(-\frac{3\pi}{4}\right)^2 = \frac{\pi^3}{32} + \frac{3\pi}{2}\cdot\frac{9\pi^2}{16} = \frac{7\pi^3}{8}.$$

51. $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x\cot^{-1}x$

$$= \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1}x\right)$$

$$= \frac{\pi^2}{4} + 2\tan^{-1}x - \pi\tan^{-1}x = \frac{\pi^2}{4} + 2\left((\tan^{-1}x)^2 - \frac{\pi}{2}\tan^{-1}x\right)$$

$$= \frac{\pi^2}{4} + 2\left[\left(\tan^{-1}x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right] = \frac{\pi^2}{4} - \frac{\pi^2}{8} + 2\left(\tan^{-1}x - \frac{\pi}{4}\right)^2$$

$$= \frac{\pi^2}{8} + 2\left(\tan^{-1}x - \frac{\pi}{4}\right)^2 \quad \text{but } \left(\tan^{-1}x - \frac{\pi}{4}\right)^2 \geq 0$$

$\therefore$ Minimum value $= \dfrac{\pi^2}{8}$.

52. We know that $0 \leq \cos^{-1}x \leq \pi$, so max. value of $\cos^{-1}x = \pi$.

$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

$\Rightarrow \cos^{-1}x = \pi,\ \cos^{-1}y = \pi,\ \cos^{-1}z = \pi$

$\Rightarrow x = \cos\pi,\ y = \cos\pi,\ z = \cos\pi$

$\Rightarrow x = -1,\ y = -1,\ z = -1.$

$\therefore x^{100} + y^{100} + z^{100} - 3 = (-1)^{100} + (-1)^{100} + (-1)^{100} - 3$

$$= 1 + 1 + 1 - 3 = 0.$$

53. As $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$, so max. value of $\sin^{-1}x = \frac{\pi}{2}$.

$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2}$

$\Rightarrow \sin^{-1}x = \frac{\pi}{2},\ \sin^{-1}y = \frac{\pi}{2},\ \sin^{-1}z = \frac{\pi}{2}$

$\Rightarrow x = \sin\frac{\pi}{2},\ y = \sin\frac{\pi}{2},\ z = \sin\frac{\pi}{2} \Rightarrow x = 1, y = 1, z = 1.$

$\therefore x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}} = 1 + 1 + 1 - \dfrac{9}{1+1+1} = 3 - 3 = 0.$

54 (i) $A = (\tan^{-1}x + \cot^{-1}x)\left((\tan^{-1}x)^2 + (\cot^{-1}x)^2 - \tan^{-1}x\cot^{-1}x\right)$

$= \frac{\pi}{2}\left[(\tan^{-1}x + \cot^{-1}x)^2 - 3\tan^{-1}x\cot^{-1}x\right]$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{2}\right)^2 - 3\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1}x\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi^2}{4} + 3\left((\tan^{-1}x)^2 - \frac{\pi}{2}\tan^{-1}x\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi^2}{4} + 3\left(\left(\tan^{-1}x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi^2}{16} + 3\left(\tan^{-1}x - \frac{\pi}{4}\right)^2\right]$

Minimum value of $A = \frac{\pi}{2}\left[\frac{\pi^2}{16} + 0\right]$ $\quad \left(\because \left(\tan^{-1}x - \frac{\pi}{4}\right)^2 \geq 0\right)$

$= \frac{\pi^3}{32}$

As $x > 0$, $0 < \tan^{-1}x < \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} < \tan^{-1}x - \frac{\pi}{4} < \frac{\pi}{4}$

$\Rightarrow 0 \leq \left(\tan^{-1}x - \frac{\pi}{4}\right)^2 \leq \frac{\pi^2}{16}$

$\therefore$ Maximum value of $A = \frac{\pi}{2}\left[\frac{\pi^2}{16} + 3\cdot\frac{\pi^2}{16}\right] = \frac{\pi^3}{8}$

$\therefore A \in \left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right)$

(ii) $B = (\cos^{-1}t + \sin^{-1}t)^2 - 2\cos^{-1}t\sin^{-1}t$

$= \left(\frac{\pi}{2}\right)^2 - 2\sin^{-1}t\left(\frac{\pi}{2} - \sin^{-1}t\right)$

$= \frac{\pi^2}{4} + 2\left((\sin^{-1}t)^2 - \frac{\pi}{2}\sin^{-1}t\right)$

$= \frac{\pi^2}{4} + 2\left[\left(\sin^{-1}t - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right] = \frac{\pi^2}{8} + 2\left(\sin^{-1}t - \frac{\pi}{4}\right)^2$

As $t \in \left[0, \frac{1}{\sqrt{2}}\right]$, $0 \leq \sin^{-1}t \leq \frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} \leq \sin^{-1}t - \frac{\pi}{4} \leq 0 \Rightarrow 0 \leq \left(\sin^{-1}t - \frac{\pi}{4}\right)^2 \leq \frac{\pi^2}{16}$.

$\therefore$ Maximum value of $\left(\sin^{-1}t - \frac{\pi}{4}\right)^2$ is attained at $\sin^{-1}t = 0$.

$\therefore$ Maximum value of $B = \frac{\pi^2}{8} + 2\cdot\frac{\pi^2}{16} = \frac{\pi^2}{8} + \frac{\pi^2}{8} = \frac{\pi^2}{4}$.

(iii) Here m = least value of $A = \frac{\pi^3}{32}$ and

M = maximum value of $B = \frac{\pi^2}{4}$.

$\therefore \frac{m - \pi M}{M} = \dfrac{\frac{\pi^3}{32} - \pi\times\frac{\pi^2}{4}}{\frac{\pi^2}{4}} = -\frac{7\pi^3}{32}\times\frac{4}{\pi^2} = -\frac{7\pi}{8}$.

$\therefore \cot^{-1}\left(\cot\left(-\frac{7\pi}{8}\right)\right) = \cot^{-1}\left(-\cot\frac{7\pi}{8}\right) = \cot^{-1}\left(\cot\left(\pi - \frac{\pi}{8}\right)\right)$

$= \cot^{-1}\left(\cot\frac{\pi}{8}\right) = \frac{\pi}{8}$.

55. $3 = 1+1+1^2, \quad 7 = 1+2+2^2, \quad 13 = 1+3+3^2, \ldots$

$$T_n = \tan^{-1}\frac{1}{1+n+n^2} = \tan^{-1}\frac{(n+1)-n}{1+n(n+1)} = \tan^{-1}(n+1) - \tan^{-1}n$$

$T_1 = \tan^{-1}2 - \tan^{-1}1$

$T_2 = \tan^{-1}3 - \tan^{-1}2$

$T_3 = \tan^{-1}4 - \tan^{-1}3$

$= \; = \; = \; =$

$T_n = \tan^{-1}(n+1) - \tan^{-1}n$

$\therefore S_n = \tan^{-1}(n+1) - \tan^{-1}1 \Rightarrow S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

56. $T_n = \tan^{-1}\frac{1}{2n^2} = \tan^{-1}\frac{2}{4n^2} = \tan^{-1}\frac{2}{1+(4n^2-1)}$

$$= \tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n-1)(2n+1)} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$T_1 = \tan^{-1}3 - \tan^{-1}1$

$T_2 = \tan^{-1}5 - \tan^{-1}3$

$T_3 = \tan^{-1}7 - \tan^{-1}5$

$- \; - \; - \; -$

$T_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$\Rightarrow S_n = \tan^{-1}(2n+1) - \tan^{-1}1 \Rightarrow S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

57. $T_n = \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} = \tan^{-1}\frac{2^{n-1}(2-1)}{1+2^{n-1}.2^n} = \tan^{-1}\frac{2^n - 2^{n-1}}{1+2^n.2^{n-1}}$

$= \tan^{-1}2^n - \tan^{-1}2^{n-1}$.

$T_1 = \tan^{-1}2 - \tan^{-1}1$

$T_2 = \tan^{-1}2^2 - \tan^{-1}2$

$T_3 = \tan^{-1}2^3 - \tan^{-1}2^2$

$- \; - \; - \; - \; - \; -$

$T_n = \tan^{-1}2^n - \tan^{-1}2^{n-1}$

$\Rightarrow S_n = \tan^{-1}2^n - \tan^{-1}1 \Rightarrow S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

58. $T_n = \cot^{-1}\left(2^{n+1} + \frac{1}{2^n}\right) = \cot^{-1}\frac{2^{n+1}2^n+1}{2^n} = \tan^{-1}\frac{2^n}{1+2^n.2^{n+1}}$

$= \tan^{-1}\frac{2^{n+1}-2^n}{1+2^{n+1}.2^n} = \tan^{-1}2^{n+1} - \tan^{-1}2^n$

$T_1 = \tan^{-1}2^2 - \tan^{-1}2$

$T_2 = \tan^{-1}2^3 - \tan^{-1}2^2$

$- \; - \; - \; -$

$T_n = \tan^{-1}2^{n+1} - \tan^{-1}2^n \; \cancel{\Rightarrow T_n - \tan^{-1}}$

$\Rightarrow S_n = \tan^{-1} 2^{n+1} - \tan^{-1} 2 \Rightarrow S_\infty = \frac{\pi}{2} - \tan^{-1} 2 = \cot^{-1} 2$

59. As $2 + r^2 + r^4 = 1 + (r^2+1)^2 - r^2 = 1 + (r^2+r+1)(r^2-r+1)$,

$\therefore T_r = \tan^{-1} \frac{2r}{2+r^2+r^4} = \tan^{-1} \frac{(r^2+r+1)-(r^2-r+1)}{1+(r^2+r+1)(r^2-r+1)}$

$= \tan^{-1}(r^2+r+1) - \tan^{-1}(r^2-r+1)$.

$T_1 = \tan^{-1} 3 - \tan^{-1} 1$

$T_2 = \tan^{-1} 7 - \tan^{-1} 3$

– – – – – –

$T_n = \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$

$\Rightarrow S_n = \tan^{-1}(n^2+n+1) - \tan^{-1} 1 = \tan^{-1}(n^2+n+1) - \frac{\pi}{4}$.

Also $S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

60. $T_r = \tan^{-1} \frac{1}{r^2-r+1} = \tan^{-1} \frac{r-(r-1)}{1+r(r-1)} = \tan^{-1} r - \tan^{-1}(r-1)$

$T_1 = \tan^{-1} 1 - \tan^{-1} 0$

$T_2 = \tan^{-1} 2 - \tan^{-1} 1$

– – – – –

$T_n = \tan^{-1} n - \tan^{-1}(n-1)$

$\Rightarrow S_n = \tan^{-1} n - \tan^{-1} 0 = \tan^{-1} n - 0 = \tan^{-1} n$.

61. $\operatorname{cosec}^{-1}\sqrt{5} + \operatorname{cosec}^{-1}\sqrt{65} + \operatorname{cosec}^{-1}\sqrt{325} + \dots$ to ∞

$(\sqrt{5} = \sqrt{1+(2.1)^2}, \sqrt{65} = \sqrt{1+(2.2^2)^2}, \sqrt{325} = \sqrt{1+(2.3^2)^2}, \dots\dots)$

$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \dots$ to ∞

$T_n = \tan^{-1}\frac{1}{2n^2}$ ... (See Q. 56)

62. $T_n = \operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)} = \operatorname{cosec}^{-1}\sqrt{(n^2+1)((n+1)^2+1)}$

$= \tan^{-1}\frac{1}{1+n(n+1)} = \tan^{-1}\frac{(n+1)-n}{1+(n+1)n}$

$= \tan^{-1}(n+1) - \tan^{-1} n$

$$\boxed{\begin{aligned} &\operatorname{cosec}^2\theta = (n^2+1)^2 + 2n(n^2+1) + n^2 + 1 \\ &\Rightarrow 1+\cot^2\theta = (n^2+1+n)^2 + 1 \\ &\Rightarrow \cot^2\theta = (n^2+n+1)^2 \\ &\Rightarrow \cot\theta = n^2+n+1 \\ &\Rightarrow \tan\theta = \frac{1}{n^2+n+1} \Rightarrow \theta = \tan^{-1}\frac{1}{1+n(n+1)} \end{aligned}}$$

$T_1 = \tan^{-1} 2 - \tan^{-1} 1$

$T_2 = \tan^{-1} 3 - \tan^{-1} 2$

– – – –

$T_n = \tan^{-1}(n+1) - \tan^{-1} n$

$\Rightarrow S_n = \tan^{-1}(n+1) - \tan^{-1} 1 = \tan^{-1}(n+1) - \frac{\pi}{4}$

63. $T_r = \sin^{-1} \dfrac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}} = \sin^{-1}\left(\dfrac{1}{\sqrt{r}}\cdot\sqrt{\dfrac{r}{r+1}} - \sqrt{\dfrac{r-1}{r}}\cdot\dfrac{1}{\sqrt{r+1}}\right)$

$= \sin^{-1}\left(\dfrac{1}{\sqrt{r}}\sqrt{1-\dfrac{1}{r+1}} - \sqrt{1-\dfrac{1}{r}}\cdot\dfrac{1}{\sqrt{r+1}}\right)$

(Let $\dfrac{1}{\sqrt{r}} = \sin\alpha$, then $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\dfrac{1}{r}}$;

$\dfrac{1}{\sqrt{r+1}} = \sin\beta$, then $\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\dfrac{1}{r+1}}$)

$= \sin^{-1}(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin^{-1}(\sin(\alpha-\beta))$

$= \alpha - \beta = \sin^{-1}\dfrac{1}{\sqrt{r}} - \sin^{-1}\dfrac{1}{\sqrt{r+1}}$

$T_1 = \sin^{-1} 1 - \sin^{-1}\dfrac{1}{\sqrt{2}}$

$T_2 = \sin^{-1}\dfrac{1}{\sqrt{2}} - \sin^{-1}\dfrac{1}{\sqrt{3}}$

— — — — — .

$T_n = \sin^{-1}\dfrac{1}{\sqrt{n}} - \sin^{-1}\dfrac{1}{\sqrt{n+1}}$

$\Rightarrow S_n = \sin^{-1} 1 - \sin^{-1}\dfrac{1}{\sqrt{n+1}} = \dfrac{\pi}{2} - \sin^{-1}\dfrac{1}{\sqrt{n+1}} = \cos^{-1}\dfrac{1}{\sqrt{n+1}}$

$\Rightarrow S_n = \tan^{-1}\sqrt{n}$.

64. Given $\cot^{-1}\left(\dfrac{n}{\pi}\right) > \dfrac{\pi}{6} \Rightarrow \cot\left(\cot^{-1}\left(\dfrac{n}{\pi}\right)\right) < \cot\dfrac{\pi}{6}$

($\because$ $\cot x$ is a decreasing function in $(0,\pi)$)

$\Rightarrow \dfrac{n}{\pi} < \sqrt{3} \Rightarrow n < \pi\sqrt{3} \Rightarrow n < 5.43$

$\Rightarrow$ maximum value of n is 5.

65.(i) Given $\tan^{-1}x : \tan^{-1}y = 1:4 \Rightarrow \tan^{-1}y = 4\tan^{-1}x$

$\Rightarrow \tan^{-1}y = 2(2\tan^{-1}x) = 2\tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{2\cdot\frac{2x}{1-x^2}}{1-\left(\frac{2x}{1-x^2}\right)^2}$

$\Rightarrow \tan^{-1}y = \tan^{-1}\dfrac{4x(1-x^2)}{x^4-6x^2+1} \Rightarrow y = \dfrac{4x(1-x^2)}{x^4-6x^2+1}$.

(ii) Let $\tan^{-1}x = \dfrac{\pi}{8} \Rightarrow 4\tan^{-1}x = \dfrac{\pi}{2}$ $\quad \left(x = \tan\dfrac{\pi}{8}\right)$

$\Rightarrow \tan^{-1}y = \dfrac{\pi}{2} \Rightarrow y \to \infty \Rightarrow x^4 - 6x^2 + 1 = 0$.

Hence, $\tan\dfrac{\pi}{8}$ is a solution of the equation $x^4 - 6x^2 + 1 = 0$.

66 $T_n = \cot^{-1}\left(n^2+\frac{3}{4}\right) = \tan^{-1}\frac{1}{n^2+\frac{3}{4}} = \tan^{-1}\frac{1}{1+\left(n^2-\frac{1}{4}\right)}$

$= \tan^{-1}\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} = \tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$

$\therefore T_1 = \tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{2}$

$T_2 = \tan^{-1}\frac{5}{2} - \tan^{-1}\frac{3}{2}$

$- \quad - \quad - \quad - \quad -$

$T_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$

$\Rightarrow S_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\frac{1}{2} \Rightarrow S_\infty = \frac{\pi}{2} - \cot^{-1}2 = \tan^{-1}2.$

67. We know that

$$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x, & |x|\le 1 \\ \pi - 2\tan^{-1}x, & x \ge 1 \\ -(\pi + 2\tan^{-1}x), & x \le -1 \end{cases}$$

$\therefore\ 2\tan^{-1}x + (\pi - 2\tan^{-1}x) = \pi$ when $x \ge 1$

Also $2\tan^{-1}x - (\pi + 2\tan^{-1}x) = -\pi$ when $x \le -1$

Hence, options (a) and (c) are correct.

68. (i) $\sin^{-1}\frac{4x}{x^2+4} + 2\tan^{-1}\left(-\frac{x}{2}\right) = \sin^{-1}\frac{2\cdot\frac{x}{2}}{1+\left(\frac{x}{2}\right)} - 2\tan^{-1}\frac{x}{2}$

$= 2\tan^{-1}\frac{x}{2} - 2\tan^{-1}\frac{x}{2}$, provided $\left|\frac{x}{2}\right| \le$

$= 0$, which is independent of x, provided $|x| \le 2$

i.e. $x \in [-2, 2]$

$\therefore$ option (c) is correct

(ii) $(x-1)(x^2+1) > 0 \Rightarrow x-1 > 0 \Rightarrow x > 1$

Now $\tan^{-1}\frac{2x}{1-x^2} = -\pi + 2\tan^{-1}x$ for $x > 1$.

$\therefore$ For $x > 1$, $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} - \tan^{-1}x\right)$

$= \sin\left(\frac{1}{2}(-\pi + 2\tan^{-1}x) - \tan^{-1}x\right) = \sin\left(-\frac{\pi}{2}\right)$

$= -\sin\frac{\pi}{2} = -1.$

$\therefore$ option (a) is correct.

(iii) $\cos^{-1}\frac{6x}{1+9x^2} = \frac{\pi}{2} - \sin^{-1}\frac{6x}{1+9x^2} = \frac{\pi}{2} - \sin^{-1}\frac{2(3x)}{1+(3x)^2}$

$$= \frac{\pi}{2} - (\pi - 2\tan^{-1}3x) \quad \text{if } 3x>1$$

$$= -\frac{\pi}{2} + 2\tan^{-1}3x, \text{ if } x>\frac{1}{3} \text{ i.e. if } x\in\left(\frac{1}{3},\infty\right)$$

$\therefore$ option (c) is correct.

69. $\tan^{-1}\frac{1}{2x+1} + \tan^{-1}\frac{1}{4x+1} = \tan^{-1}\frac{2}{x^2}, \quad x\neq 0$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{2x+1}\cdot\frac{1}{4x+1}}\right) = \tan^{-1}\frac{2}{x^2}$$

$$\Rightarrow \frac{6x+2}{8x^2+6x} = \frac{2}{x^2} \Rightarrow \frac{3x+1}{4x^2+3x} = \frac{2}{x^2}$$

$$\Rightarrow 3x^3-7x^2-6x=0 \Rightarrow 3x^2-7x-6=0 \quad (\because x\neq 0)$$

$$\Rightarrow (x-3)(3x+2)=0 \Rightarrow x = 3, -\frac{2}{3}.$$

But $x=-\frac{2}{3}$ does not satisfy the given equation (check),

So $x=3$ is the only solution of the given equation

$\therefore$ option (a) is correct.

70. we know that $0\le\{x\}<1 \Rightarrow 0\ge -\{x\} > -1$

$$\Rightarrow -1 < -\{x\} \le 0 \Rightarrow \frac{\pi}{2} \le \cos^{-1}(-\{x\}) < \pi$$

(Note that $\cos^{-1}x$ is a decreasing function in $[0,\pi]$)

$\Rightarrow x\in\left[\frac{\pi}{2},\pi\right) \Rightarrow$ option (c) is correct.

71. (a) $\sqrt{2}(\sin 2x-\cos 2x) = 2\cdot\left(\frac{1}{\sqrt{2}}\sin 2x - \frac{1}{\sqrt{2}}\cos 2x\right)$

$$= 2\left(\sin 2x\cos\frac{\pi}{4} - \cos 2x\sin\frac{\pi}{4}\right) = 2\sin\left(2x-\frac{\pi}{4}\right).$$

As $-1\le \sin\left(x-\frac{\pi}{4}\right) \le 1$ for all $x\in R$,

$$-2 \le 2\sin\left(x-\frac{\pi}{4}\right) \le 2$$

$\Rightarrow$ Maximum value $=2$ and minimum value $=-2$

$\therefore$ the absolute difference of greatest and least value of $\sqrt{2}(\sin 2x-\cos 2x) = |2-(-2)| = 4.$

$\Rightarrow (a)\leftrightarrow(r).$

(b) Let $f(x) = x^2 - 4x + 3 = (x-2)^2 - 1$, $x \in [1, 3]$

Least value of $f(x) = -1$ (when $x = 2$)

Greatest value of $f(x) = 0$ (when $x = 1$ or $x = 3$)

$\therefore$ Absolute difference of greatest value and least value of $f(x)$

$$= |0 - (-1)| = 1$$

$$\Rightarrow (b) \leftrightarrow (s)$$

(c) Let $x = \tan\alpha \Rightarrow \alpha = \tan^{-1}x$

$$f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\alpha}{1 + \tan\frac{\pi}{4}\tan\alpha}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \alpha\right)\right)$$

$$= \frac{\pi}{4} - \alpha = \frac{\pi}{4} - \tan^{-1}x.$$

$x \in [0, 1] \Rightarrow 0 \le \tan^{-1}x \le \frac{\pi}{4} \Rightarrow 0 \ge -\tan^{-1}x \ge -\frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} \le -\tan^{-1}x \le 0 \Rightarrow 0 \le \frac{\pi}{4} - \tan^{-1}x \le \frac{\pi}{4}.$

Greatest value of $f(x) = \frac{\pi}{4}$.

$\therefore$ (c) $\leftrightarrow$ (p)

(d) $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \Rightarrow 0 \le x^2 \le \frac{1}{2} \Rightarrow \frac{\pi}{3} \le \cos^{-1}x^2 \le \frac{\pi}{2}.$

$\Rightarrow$ greatest value $= \frac{\pi}{2}$ and least value $= \frac{\pi}{3}$.

$\therefore$ Absolute difference of greatest value and least value

$$= \left|\frac{\pi}{2} - \frac{\pi}{3}\right| = \frac{\pi}{6}$$

$\therefore$ (d) $\leftrightarrow$ (q).

72. (a) Statement II is true i.e. $\sin^{-1}\frac{2x}{1+x^2} = \pi - 2\tan^{-1}x, \ \forall x > 1$

$\Rightarrow f(x) = \pi - 2\tan^{-1}x, \ \forall x > 1$

$\therefore f'(x) = 0 - 2 \cdot \frac{1}{1+x^2} \Rightarrow f'(2) = -2 \cdot \frac{1}{1+2^2} = -\frac{2}{5}.$

It follow that if statement II is correct then statement is also correct i.e. if statement II is correct then statement I is also true.

Hence, (a) is correct option.

Note that (b), (c) and (d) are all incorrect.