Handwritten notes and problems

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Topic - Trigonometric Equations

Trigonometric equations

Trigonometric equation — An equation involving trigonometric functions of an unknown real number is called a trigonometric equation. A real number that satisfies the equation is called a solution of the equation. The solutions satisfying $0 \le x < 2\pi$ are called principal solutions.

1. (i) $\sin x = 0 \Rightarrow x = n\pi,\ n \in I$; $\sin x = 1 \Rightarrow x = 2n\pi + \frac{\pi}{2}$

(ii) $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2},\ n \in I$; $\cos x = 1 \Rightarrow x = 2n\pi$

(iii) $\tan x = 0 \Rightarrow x = n\pi,\ n \in I$; $\tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$

2. (i) $\sin x = \sin \alpha \Rightarrow x = n\pi + (-1)^n \alpha,\ n \in I$

(ii) $\cos x = \cos \alpha \Rightarrow x = 2n\pi \pm \alpha,\ n \in I$

(iii) $\tan x = \tan \alpha \Rightarrow x = n\pi + \alpha,\ n \in I$

3. (i) $\sin^2 x = \sin^2 \alpha \Rightarrow x = n\pi \pm \alpha,\ n \in I$

(ii) $\cos^2 x = \cos^2 \alpha \Rightarrow x = n\pi \pm \alpha,\ n \in I$

(iii) $\tan^2 x = \tan^2 \alpha \Rightarrow x = n\pi \pm \alpha,\ n \in I$

4. $b \sin x + a \cos x = c$, where $|c| \le \sqrt{a^2+b^2}$

$$\Rightarrow \frac{b}{\sqrt{a^2+b^2}} \sin x + \frac{a}{\sqrt{a^2+b^2}} \cos x = \frac{c}{\sqrt{a^2+b^2}}$$

Find a real number β $(0 \le \beta < 2\pi)$ such that $\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$;

find a real number α $(0 \le \alpha < 2\pi)$ such that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

Given equation can be written as

$$\cos x \cos\alpha + \sin x \sin\alpha = \cos\beta \Rightarrow \cos(x-\alpha) = \cos\beta$$

$$\Rightarrow x - \alpha = 2n\pi \pm \beta$$

$$\cancel{\Rightarrow x = 2n\pi + \beta}$$

$$\Rightarrow x = 2n\pi + \alpha \pm \beta,\ n \in I$$

5. (i) For solving $\sin x = c$, $|c| \le 1$,

find smallest number $\alpha \in [0, 2\pi)$ such that $\sin\alpha = c$.

$\therefore \sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n\alpha,\ n \in I$

(ii) For solving $\cos x = c$, $|c| \le 1$,

find smallest number $\alpha \in [0, 2\pi)$ such that $\cos\alpha = c$

$\therefore \cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha,\ n \in I$

(iii) For solving $\tan x = c$, $c \in R$,

find smallest number $\alpha \in [0, 2\pi)$ such that $\tan\alpha = c$

$\therefore \tan x = \tan\alpha \Rightarrow x = n\pi + \alpha,\ n \in I$.

1. The general solution of equation $\sin^2\theta \sec\theta + \sqrt{3}\tan\theta = 0$ is

(a) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}$ (b) $\theta = n\pi$ UD

(c) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{6}$ (d) $\theta = \frac{n\pi}{2}$

2. If $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$, then θ equals UD

(a) $\frac{n\pi}{2} + \frac{\pi}{6}$ (b) $\frac{n\pi}{2} - \frac{\pi}{6}$ (c) $\frac{n\pi}{2} + \frac{\pi}{3}$ (d) $\frac{n\pi}{2} - \frac{\pi}{3}$

3. The most general solution of $\sec x - 1 = (\sqrt{2}-1)\tan x$ is UD

(a) $n\pi + \frac{\pi}{8}$ (b) $2n\pi,\ 2n\pi + \frac{\pi}{4}$ (c) $2n\pi$ (d) none of these

4. The general solution of

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is UD

(a) $n\pi + \frac{\pi}{8}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$ (c) $(-1)^n\left(\frac{n\pi}{2}\right) + \frac{\pi}{8}$ (d) $2n\pi + \cot^{-1}\frac{3}{2}$

5. The values of x such that $-\pi < x < \pi$ and satisfying the equation are given by $8^{1+|\cos x|+|\cos^2 x|+|\cos^3 x|+\cdots} = 4^3$ equals

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$ UD

* 6. The solution of the equation $4\sin^4 x + \cos^4 x = 1$ is UD

(a) $x = 2n\pi$ (b) $x = n\pi + 1$ (c) $x = (n+2)\pi$ (d) none of these

7. The number of points of intersection of $2y = 1$ and $y = \cos x$ in

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is UD

(a) 1 (b) 2 (c) 3 (d) 4

8. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is

(a) 6, (b) 1 (c) 2 (d) 4 UD

9. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is UD

(a) 0 (b) 5 (c) 6 (d) 10

10. The number of values of x in $[0, 5\pi]$ satisfying

$3\cos 2x - 10\cos x + 7 = 0$ is UD

(a) 5 (b) 6 (c) 8 (d) 10

10

11. The ~~sum~~ number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3 UD

12. If $0 \le x \le \pi$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{4}$ UD

13. The number of solutions of $16^{\sin^2 x} + 16^{\cos^2 x} = 10$ in $[0, 2\pi]$ is

(a) 8 (b) 6 (c) 4 (d) 2 UD

14. The number of solutions of the equation $3x + 2\tan x = \frac{5\pi}{2}$ in $x \in [0, 2\pi]$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4 UD

15. The number of solutions of $\cos x = |1 + \sin x|$ for $x \in [0, 3\pi]$ is

(a) 3 (b) 2 (c) 4 (d) none of these UD

16. If $\sin^2 x - 2\sin x - 1 = 0$ is to be satisfied for exactly 4 distinct values of $x \in [0, n\pi]$, $\forall n \in \underline{N}$, then the least value of n is

(a) 2 (b) 6 (c) 4 (d) 8 UD

* 17. The set of values of x satisfying the inequation

$2\sin^2 x - 5\sin x + 2 > 0$ where $0 < x < 2\pi$ is UD

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (b) $\left[0, \frac{\pi}{6}\right] \cap \left[\frac{5\pi}{6}, 2\pi\right]$

(c) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \pi\right]$ (d) none of these

18. The number of solutions of $\sin x = \frac{|x|}{10}$ is UD

(a) 4 (b) 6 (c) 8 (d) none of these

19. The equation $a\sin x + b\cos x = c$, where $|c| > \sqrt{a^2 + b^2}$ has

(a) one solution (b) two solutions

(c) no solution (d) infinite number of solutions UD.

20. The number of distinct solutions of

$\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta$ in $\left[0, \frac{\pi}{2}\right]$ is

UD

(a) 4 (b) 5 (c) 8 (d) 9

21. The sum of all solutions of $\cos x \cos(\frac{\pi}{3}+x) \cos(\frac{\pi}{3}-x) = \frac{1}{4}$, !!
$x \in [0, 6\pi]$ is
UD
(a) 15π (b) 30π (c) $\frac{110\pi}{3}$ (d) none of these

22. The general solution of the equation $\sum_{r=1}^{n} \cos(r^2 x) \sin(r x) = \frac{1}{2}$ is

(a) $2m\pi + \frac{\pi}{6}, m \in I$ ✓(b) $\frac{4m+1}{n(n+1)} \cdot \frac{\pi}{2}, m \in I$ UD

(c) $\frac{4m-1}{n(n+1)} \cdot \frac{\pi}{2}, m \in I$ (d) none of these

23. The number of solutions of the equation $\sin x + 2\sin 2x = 3 + \sin 3x$ in the interval $[0, \pi]$ is
UD
✓(a) 0 (b) 1 (b) 2 (c) 3

* 24. The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k+1$ has a ~~unique~~ solution is
UD
(a) 4 ✓(b) 8 (c) 10 (d) 12

25. If $[\sin x] + [\sqrt{2}\cos x] = -3$, $x \in [0, 2\pi]$, where $[x]$ denotes the greatest integer function, then x belongs to UD
✓(a) $(\pi, \frac{5\pi}{4})$ (b) $[\pi, \frac{5\pi}{4}]$ (c) $(\frac{5\pi}{4}, 2\pi)$ (d) $[\frac{5\pi}{4}, 2\pi]$

26. If $\cos 2\theta = (\sqrt{2}+1)(\cos\theta - \frac{1}{\sqrt{2}})$, then θ is equal to
UD
(a) $2n\pi$ ✓(b) $2n\pi \pm \frac{\pi}{4}$ ✓(c) $2n\pi \pm \frac{\pi}{3}$ (d) none of these

27. If $\frac{1}{6}\sin x$, $\cos x$ and $\tan x$ are in G.P., then x equals
✓(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{6}$ (c) $n\pi + (-1)^n \frac{\pi}{3}$ (d) $n\pi + \frac{\pi}{3}$ UD

28. The number of values of x satisfying $|\sqrt{3}\cos x - \sin x| \geq 2$ is
(a) 0 (b) 2 ✓(c) 4 (d) 8 UD

* 29. If $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$, then the possible values of x are
(a) $n\pi$ or $n\pi + (-1)^n \frac{\pi}{6}$ (b) $n\pi$ or $n\pi + (-1)^n \frac{\pi}{3}$ UD
✓(c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) $n\pi$

Solutions — Trigonometric equations

1. $\sin^2\theta \sec\theta + \sqrt{3}\tan\theta = 0 \Rightarrow \sin\theta \cdot \frac{\sin\theta}{\cos\theta} + \sqrt{3}\tan\theta = 0$

$\Rightarrow \sin\theta\tan\theta + \sqrt{3}\tan\theta = 0 \Rightarrow \tan\theta(\sin\theta + \sqrt{3}) = 0$

$\Rightarrow \tan\theta = 0$ or $\sin\theta = -\sqrt{3}$ rejected (not possible)

$\Rightarrow \theta = n\pi,\ n \in I$

2. $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$

$\Rightarrow 3(\cos^2\theta - \sin^2\theta) - \sqrt{3}(2\sin\theta\cos\theta) = 0$

$\Rightarrow 3\cos 2\theta - \sqrt{3}\sin 2\theta = 0 \Rightarrow \tan 2\theta = \sqrt{3} \Rightarrow \tan 2\theta = \tan\frac{\pi}{3}$

$\Rightarrow 2\theta = n\pi + \frac{\pi}{3} \Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{6}$.

3. $\sec x - 1 = (\sqrt{2}-1)\tan x \Rightarrow \frac{1}{\cos x} - 1 = (\sqrt{2}-1)\frac{\sin x}{\cos x}$

$\Rightarrow 1 - \cos x = (\sqrt{2}-1)\sin x \Rightarrow 2\sin^2\frac{x}{2} = (\sqrt{2}-1).2\sin\frac{x}{2}\cos\frac{x}{2}$

$\Rightarrow 2\sin\frac{x}{2}\left(\sin\frac{x}{2} - (\sqrt{2}-1)\cos\frac{x}{2}\right) = 0$

$\Rightarrow \sin\frac{x}{2} = 0$ or $\tan\frac{x}{2} = \sqrt{2}-1$ i.e. $\tan\frac{x}{2} = \tan\frac{\pi}{8}$

$\Rightarrow \frac{x}{2} = n\pi$ or $\frac{x}{2} = n\pi + \frac{\pi}{8} \Rightarrow x = 2n\pi,\ 2n\pi + \frac{\pi}{4}$.

4. $(\sin 3x + \sin x) - 3\sin 2x = (\cos 3x + \cos x) - 3\cos 2x$

$\Rightarrow 2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cos x - 3\cos 2x$

$\Rightarrow \sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$

$\Rightarrow (2\cos x - 3)(\sin 2x - \cos 2x) = 0$

$\Rightarrow \cos x = \frac{3}{2}$ (not possible), $\tan 2x = 1$ i.e. $\tan 2x = \tan\frac{\pi}{4}$

$\Rightarrow 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}$.

5. $|\cos^2 x| = |\cos x|^2,\ |\cos^3 x| = |\cos x|^3 \ldots$

$1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \ldots \text{to } \infty = 1 + |\cos x| + |\cos x|^2 + |\cos x|^3 + \ldots \text{to } \infty$ G.P.

$= \frac{1}{1 - |\cos x|}$.

$\therefore 8^{1 + |\cos x| + |\cos^2 x| + \ldots \text{to } \infty} = 4^3 \Rightarrow 8^{\frac{1}{1-|\cos x|}} = 64 = 8^2$

$\Rightarrow \frac{1}{1-|\cos x|} = 2 \Rightarrow 1 - |\cos x| = \frac{1}{2} \Rightarrow |\cos x| = \frac{1}{2}$

$\Rightarrow \cos x = \frac{1}{2}, -\frac{1}{2} \Rightarrow x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$ $\quad x \in (-\pi, \pi)$

6. $4\sin^4 x + \cos^4 x - 1 = 0 \Rightarrow 4\sin^4 x + (\cos^2 x - 1)(\cos^2 x + 1) = 0$

$\Rightarrow 4\sin^4 x - \sin^2 x(\cos^2 x + 1) = 0$

$\Rightarrow \sin^2 x(4\sin^2 x - \cos^2 x - 1) = 0 \Rightarrow \sin^2 x(5\sin^2 x - 2) = 0$

$\Rightarrow \sin^2 x = 0$ or $\sin^2 x = \frac{2}{5}$

$\Rightarrow \sin^2 x = \sin^2 0$ or $\sin^2 x = \sin^2\alpha$, where $\sin\alpha = \pm\sqrt{\frac{2}{5}}$

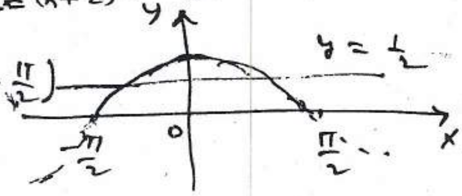
$\Rightarrow x = n\pi$ or $x = n\pi \pm \alpha$

$\Rightarrow x = (n+2)\pi$ ($\because x = n\pi$ and $x = (n+2)\pi$ give the same solutions)

7. $2y = 1$ i.e. $y = \frac{1}{2}$, $y = \cos x \cdot (-\frac{\pi}{2} \le x \le \frac{\pi}{2})$

Draw their graphs

There are two points of intersection

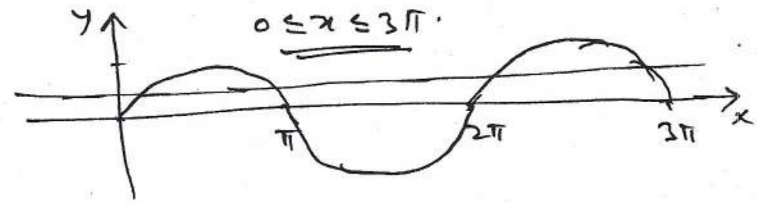


8. $2\sin^2 x + 5\sin x - 3 = 0 \Rightarrow (2\sin x - 1)(\sin x + 3) = 0$

$\Rightarrow \sin x = \frac{1}{2}$, $\sin x = -3$ (not possible).

$y = \sin x$ and $y = \frac{1}{2}$

There are 4 points of intersection

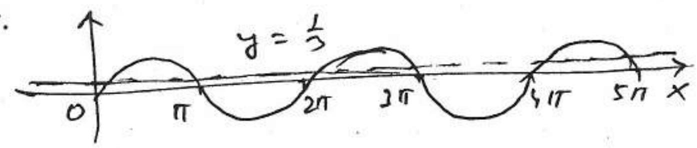


9. $3\sin^2 x - 7\sin x + 2 = 0 \Rightarrow (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow \sin x = \frac{1}{3}$, $\sin x = 2$ (not possible)

$y = \sin x$ & $y = \frac{1}{3}$. $x \in [0, 5\pi)$

There are 6 points of intersection



10. $3\cos 2x - 10\cos x + 7 = 0$

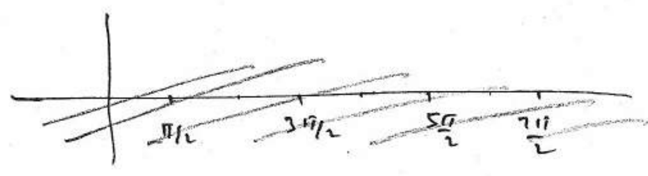
$\Rightarrow 3(2\cos^2 x - 1) - 10\cos x + 7 = 0$

$\Rightarrow 6\cos^2 x - 10\cos x + 4 = 0$

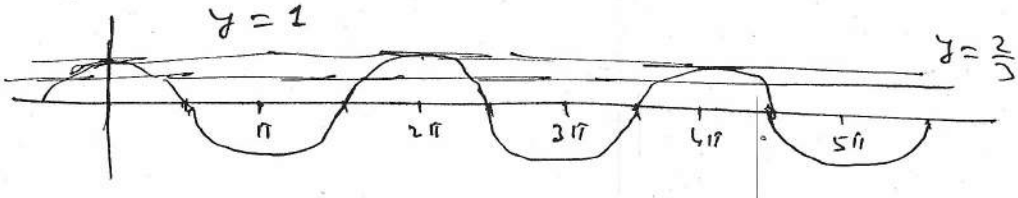
$\Rightarrow 3\cos^2 x - 5\cos x + 2 = 0$

$\Rightarrow (3\cos x - 2)(\cos x - 1) = 0$

$\Rightarrow \cos x = \frac{2}{3}, 1$



The lines $y = \frac{2}{3}$, $y = 1$ intersect the graph of $y = \cos x$ in $[0, 5\pi]$ at 8 points



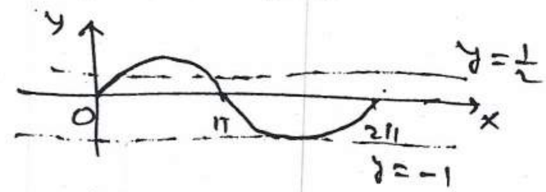
11. $\tan x + \sec x = 2\cos x \Rightarrow \frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2\cos x$

$\Rightarrow \sin x + 1 = 2\cos^2 x \Rightarrow \sin x + 1 = 2(1-\sin^2 x)$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow (2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}, -1$

The lines $y = \frac{1}{2}$, $y = -1$ and $y = \sin x$ intersect at three points in $[0, 2\pi]$



12. $81^{\sin^2 x} + 81^{\cos^2 x} = 30 \Rightarrow 81^{\sin^2 x} + 81^{1-\sin^2 x} = 30$

$\Rightarrow 81^{\sin^2 x} + \frac{81}{81^{\sin^2 x}} = 30$, let $y = 81^{\sin^2 x}$

$\Rightarrow y + \frac{81}{y} = 30 \Rightarrow y^2 - 30y + 81 = 0 \Rightarrow (y-27)(y-3) = 0$

$\Rightarrow y = 27, 3 \Rightarrow 81^{\sin^2 x} = 27, 3$

$\Rightarrow 3^{4\sin^2 x} = 3^3, 3^1 \Rightarrow 4\sin^2 x = 3, 1 \Rightarrow \sin^2 x = \frac{3}{4}, \frac{1}{4}$

$\Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$, $\sin x = \pm\frac{1}{2}$; ~~$x \in [0, 2\pi]$~~ $x \in [0, \pi]$

$\Rightarrow \sin x = \frac{\sqrt{3}}{2}$, $\sin x = \frac{1}{2}$ $\qquad \Rightarrow \sin x \geq 0$

~~The lines $y = \frac{\sqrt{3}}{2}$, $y = -\frac{\sqrt{3}}{2}$, $y = \frac{1}{2}$, $y = \frac{1}{2}$ and $y = \sin x$~~

$\Rightarrow x = \frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

~~of the lines $y = 3$~~

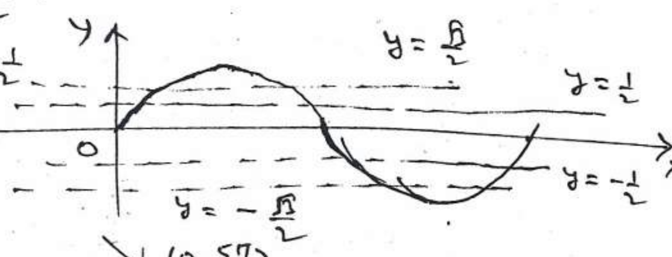
13. $16^{\sin^2 x} + 16^{1-\sin^2 x} = 10 \Rightarrow 16^{\sin^2 x} + \frac{16}{16^{\sin^2 x}} = 10$, let $y = 16^{\sin^2 x}$

$\Rightarrow y + \frac{16}{y} = 10 \Rightarrow y^2 - 10y + 16 = 0 \Rightarrow y = 8, 2 \Rightarrow 16^{\sin^2 x} = 8, 2$

$\Rightarrow 2^{4\sin^2 x} = 2^3, 2^1 \Rightarrow 4\sin^2 x = 3, 1 \Rightarrow \sin^2 x = \frac{3}{4}, \frac{1}{2}$

$\Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}$.

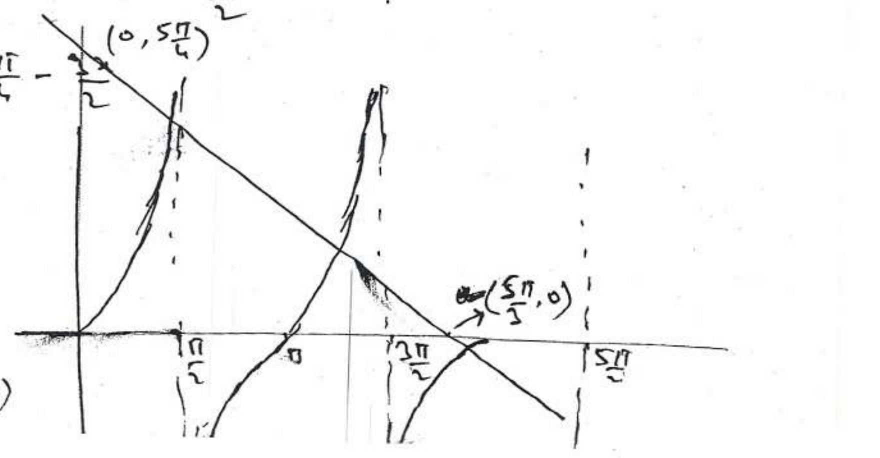
The ~~lines~~ graphs $y = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{1}{2}$ and the graph of $y = \sin x$ in $[0, 2\pi]$ intersect in 8 points.



14. $3x + 2\tan x = \frac{5\pi}{2} \Rightarrow \tan x = \frac{5\pi}{4} - \frac{3x}{2}$

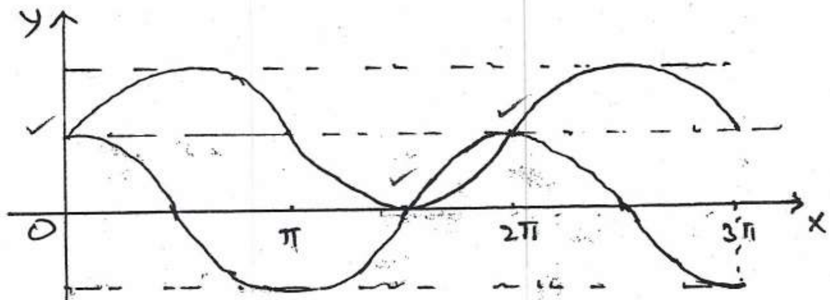
The graphs $y = \frac{5\pi}{4} - \frac{3x}{2}$ and $y = \tan x$ in $[0, 2\pi]$ intersect at three points.

The line $y = \frac{5\pi}{4} - \frac{3x}{2}$ passes through points $(0, \frac{5\pi}{4})$ and $(\frac{5\pi}{3}, 0)$



15. As $-1 \le \sin x \le 1$ for all $x \in R \Rightarrow 1+\sin x \ge 0 \Rightarrow |1+\sin x| = 1+\sin x$

The graphs of $y = \cos x$ and $y = 1 + \sin x$ in $[0, 3\pi]$ intersect in 3 points. $x = 0, \frac{3\pi}{2}, 2\pi$

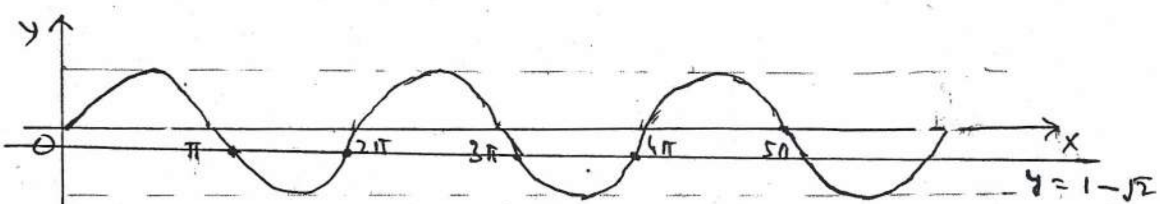


16. $\sin^2 x - 2\sin x - 1 = 0 \Rightarrow \sin x = \dfrac{2 \pm \sqrt{4 - 1.(-1)}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2}$

$= 1-\sqrt{2}$, $(1+\sqrt{2})$ not possible.

$\therefore \sin x = 1-\sqrt{2}$

Draw graphs of $y = \sin x$ and $y = 1-\sqrt{2}$ i.e. $y = -(\sqrt{2}-1)$



clearly 4 solutions lie in $[0, 4\pi]$

$\therefore$ The least value of n is 4.

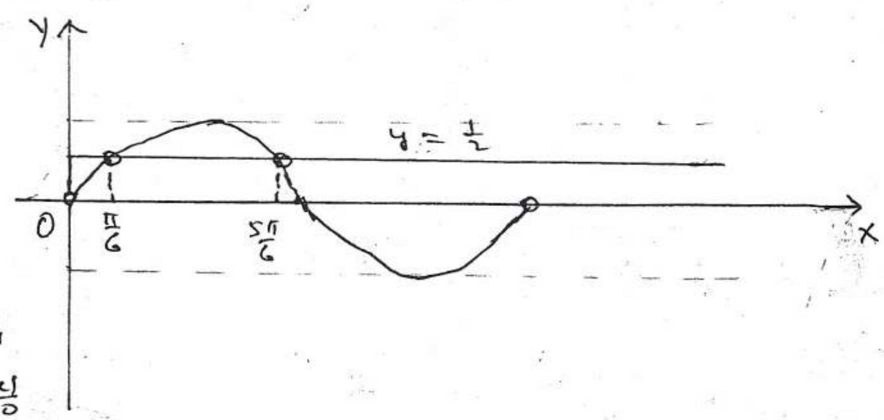
17. $2\sin^2 x - 5\sin x + 2 > 0 \Rightarrow (2\sin x - 1)(\sin x - 2) > 0$

$\Rightarrow \sin x < \frac{1}{2}$ or $\sin x > 2$ but $\sin x > 2$ is not possible.

$\Rightarrow \sin x < \frac{1}{2}$.

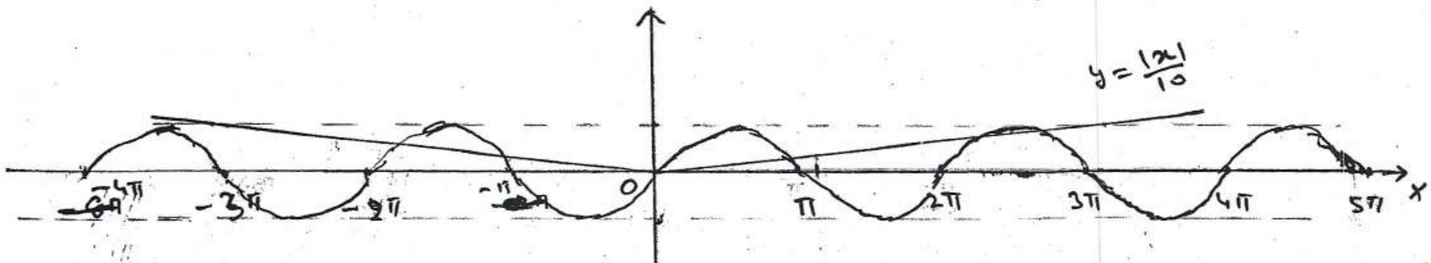
Draw graphs of $y = \frac{1}{2}$ and $y = \sin x$ in $0 < x < 2\pi$

from graphs, we find that $x \in (0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, 2\pi)$



18 Draw the graphs of $y = \sin x$ and $y = \frac{|x|}{10}$

$\because y = \frac{x}{10}$, passes through $(0,0)$, $(10,1)$



Graphs of $y = \sin x$ and $y = \frac{|x|}{10}$ meet exactly in 6 points

19. $a \sin x + b \cos x = c$ has solutions only when $|c| \le \sqrt{a^2+b^2}$.

~~So, the given~~
Here, $|c| > \sqrt{a^2+b^2}$. So, the given equation has no solution

20. $\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta \Rightarrow 2\sin 5\theta \cos 3\theta = 2 \sin 9\theta \cos 7\theta$

$\Rightarrow \sin 8\theta + \sin 2\theta = \sin 16\theta + \sin 2\theta$

$\Rightarrow \cancel{\sin 16\theta}\ \sin 16\theta - \sin 8\theta = 0 \Rightarrow 2\cos 12\theta \sin 4\theta = 0$

$\Rightarrow \cos 12\theta = 0$ or $\sin 4\theta = 0$

$\Rightarrow 12\theta = (2n+1)\frac{\pi}{2}$ or $4\theta = n\pi$

$\Rightarrow \theta = (2n+1)\frac{\pi}{24}$ or $\theta = \frac{n\pi}{4}$ $\quad \theta \in [0, \frac{\pi}{2}]$

$\Rightarrow \theta = \frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{3\pi}{8}, \frac{11\pi}{24}$ or $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$

$\therefore \theta$ has 9 values in $[0, \frac{\pi}{2}]$

21. $\cos x \cos(\frac{\pi}{3}+x) \cos(\frac{\pi}{3}-x) = \frac{1}{4} \Rightarrow \frac{1}{4}\cos 3x = \frac{1}{4}$

$\Rightarrow \cos 3x = 1 \Rightarrow 3x = 2n\pi,\ n \in I \Rightarrow x = \frac{2n\pi}{3}$ but $x \in [0, 6\pi]$

$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \frac{8\pi}{3}, \dots, \frac{18\pi}{3}$

Sum of these solutions $= \frac{2\pi}{3}(1+2+3+\dots+9)$

$= \frac{2\pi}{3} \cdot \frac{9(9+1)}{2} = 30\pi$

22. $\sum_{r=1}^{n} \cos(r^2 x)\sin(rx) = \frac{1}{2} \Rightarrow \sum_{r=1}^{n} 2\cos(r^2x)\sin(rx) = 1$

$\Rightarrow \sum_{r=1}^{n} (\sin(r^2x + rx) - \sin(r^2x - rx)) = 1$

$\Rightarrow \sum_{r=1}^{n} (\sin r(r+1)x - \sin r(r-1)x) = 1$

$\Rightarrow (\sin 2x - \sin 0) + (\sin 6x - \sin 2x) + (\sin 12x - \sin 6x)$
$\quad + \cdots + (\sin n(n+1)x - \sin n(n-1)x) = 1$

$\Rightarrow -\sin 0 + \sin n(n+1)x = 1$

$\Rightarrow \sin n(n+1)x = 1 \Rightarrow n(n+1)x = 2m\pi + \frac{\pi}{2}, m \in I$

$\Rightarrow n(n+1)x = (4m+1)\frac{\pi}{2} \Rightarrow x = \frac{4m+1}{n(n+1)} \cdot \frac{\pi}{2}, m \in I$

23. $\sin x + 2\sin 2x = 3 + \sin 3x$

$\Rightarrow (\sin 3x - \sin x) - 2\sin 2x + 3 = 0$

$\Rightarrow 2\cos 2x \sin x - 2.2\sin x \cos x + 3 = 0$

$\Rightarrow 2\sin x(\cos 2x - 2\cos x) + 3 = 0$

~~$\Rightarrow 2\sin x (2\cos^2 x - 2\cos x$~~

$\Rightarrow 2\sin x(2\cos^2 x - 1 - 2\cos x) + 3 = 0$

$\Rightarrow \sin x\,((4\cos^2 x - 4\cos x) - 2) + 3 = 0$

$\Rightarrow \sin x\,((2\cos x - 1)^2 - 3) + 3 = 0$

$\Rightarrow \sin x\,(2\cos x - 1)^2 + 3(1 - \sin x) = 0$

Now $x \in [0, \pi] \Rightarrow 0 \le x \le \pi \Rightarrow 0 \le \sin x \le 1$

$\Rightarrow 1 - \sin x \geq 0$ and $\sin x(2\cos x - 1)^2 \geq 0$)

$\therefore$ each term is equal to zero ie.

$1 - \sin x = 0$ and $\sin x(2\cos x - 1)^2 = 0$

$\Rightarrow \sin x = 1$ and $1.(2\cos x - 1)^2 = 0$

$\Rightarrow \cos x = 0$ and $(2.0 - 1)^2 = 0$ ie. $1 = 0$ (not possible)

Hence, the given equation has no solution in $[0, \pi]$

24. $7\cos x + 5\sin x = 2k + 1$

(Put $7 = r\cos\alpha$ and $5 = r\sin\alpha \Rightarrow r^2 = 74 \Rightarrow r = \sqrt{74}$)

$\Rightarrow \cos x\cos\alpha + \sin x\sin\alpha = \frac{2k+1}{\sqrt{74}}$

$\Rightarrow \cos(x - \alpha) = \frac{2k+1}{\sqrt{74}}$.

For the given equation to have a solution, $\left|\frac{2k+1}{\sqrt{74}}\right| \le 1$

$\Rightarrow -1 \le \frac{2k+1}{\sqrt{74}} \le 1 \Rightarrow -\sqrt{74} \le 2k+1 \le \sqrt{74}$ $\quad \sqrt{74} = 8.6$ (app.)

$\Rightarrow -8.6 \le 2k+1 \le 8.6 \Rightarrow -9.6 \le 2k \le 7.6$

$\Rightarrow -4.8 \le k \le 3.8$ but $k \in I$

$k = -4, -3, -2, -1, 0, 1, 2, 3$. So k has 8 integral values.

25. $[\sin x] + [\sqrt{2}\cos x] = -3$

$(-1 \le \sin x \le 1, \quad -\sqrt{2} \le \sqrt{2}\cos x \le \sqrt{2})$

$\Rightarrow [\sin x] = -1$ and $[\sqrt{2}\cos x] = -2$

$\Rightarrow -1 \le \sin x < 0$ and $-2 \le \sqrt{2}\cos x < -1$

i.e. $-\sqrt{2} \le \cos x < -\frac{1}{\sqrt{2}}$

$\Rightarrow -1 \le \sin x < 0$ and $-1 \le \cos x < -\frac{1}{\sqrt{2}}$ $(\because -1 \le \cos x \le 1)$

As $\sin x$ and $\cos x$ are both negative and $x \in [0, 2\pi)$

i.e. $\sin x$ and $\cos x$ both lie in III rd quad.

$\Rightarrow x \in \left(\pi, \frac{3\pi}{2}\right)$ and $x \in \left(\pi, \frac{5\pi}{4}\right)$

$\Rightarrow x \in \left(\pi, \frac{5\pi}{4}\right)$

26. $\cos 2\theta = (\sqrt{2}+1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right) \Rightarrow 2\cos^2\theta - 1 = (\sqrt{2}+1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow 2\left(\cos\theta - \frac{1}{\sqrt{2}}\right)\left(\cos\theta + \frac{1}{\sqrt{2}}\right) - (\sqrt{2}+1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right) = 0$

$\Rightarrow \left(\cos\theta - \frac{1}{\sqrt{2}}\right)\left(2\left(\cos\theta + \frac{1}{\sqrt{2}}\right) - (\sqrt{2}+1)\right) = 0$

$\Rightarrow \left(\cos\theta - \frac{1}{\sqrt{2}}\right)\left(2\cos\theta + \sqrt{2} - \sqrt{2} - 1\right) = 0$

$\Rightarrow \left(\cos\theta - \frac{1}{\sqrt{2}}\right)\left(2\cos\theta - 1\right) = 0$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ or $\cos\theta = \frac{1}{2}$

$\Rightarrow \cancel{\theta} \ \cos\theta = \cos\frac{\pi}{4}$ or $\cos\theta = \cos\frac{\pi}{3}$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{4}$ or $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in I$

27. Since $\frac{1}{6}\sin x$, $\cos x$, $\tan x$ are in G.P., $\cos^2 x = \frac{1}{6}\sin x \tan x$

$\Rightarrow \cos^2 x = \frac{1}{6}\sin x \cdot \frac{\sin x}{\cos x} \Rightarrow 6\cos^3 x = \sin^2 x \Rightarrow 6\cos^3 x = 1 - \cos^2 x$

$\Rightarrow 6\cos^3 x + \cos^2 x - 1 = 0 \Rightarrow (2\cos x - 1)(3\cos^2 x + 2\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2}$ or $3\cos^2 x + 2\cos x + 1 = 0$.

But $3\cos^2 x + 2\cos x + 1 = 0 \Rightarrow \cos x = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-1 \pm i\sqrt{2}}{3}$, which is not a real number, so $3\cos^2 x + 2\cos x + 1 \ne 0$.

$\therefore \cos x = \frac{1}{2} \Rightarrow \cos x = \cos\frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3}$.

28. $|\sqrt{3}\cos x - \sin x| \geq 2 \Rightarrow \left|\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right| \geq 1$

$\Rightarrow \left|\cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6}\right| \geq 1 \Rightarrow \left|\cos\left(x+\frac{\pi}{6}\right)\right| \geq 1$, but $\left|\cos\left(x+\frac{\pi}{6}\right)\right| \leq 1$ $\forall x \in \mathbb{R}$

$\Rightarrow \left|\cos\left(x+\frac{\pi}{6}\right)\right| = 1 \Rightarrow \cos\left(x+\frac{\pi}{6}\right) = 1$ or -1

$\Rightarrow \cos\left(x+\frac{\pi}{6}\right) = \cos 0$ or $\cos\left(x+\frac{\pi}{6}\right) = \cos\pi$

$\Rightarrow x+\frac{\pi}{6} = 2n\pi \pm 0$, $x+\frac{\pi}{6} = 2n\pi \pm \pi$ but $x \in [0, 4\pi]$

$\Rightarrow x+\frac{\pi}{6} = 2\pi, 4\pi \;;\; \pi, 3\pi$

$\Rightarrow \cancel{2x}\; x = 2\pi - \frac{\pi}{6}, 4\pi - \frac{\pi}{6}, \pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$

$\Rightarrow x = \frac{11\pi}{6}, \frac{23\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}$.

$\therefore$ The given inequation has 4 solutions in $[0, 4\pi]$

29. $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$, $\cos x \neq 0 \Rightarrow x \neq (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow |\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = |\cos x|^0$ $\Rightarrow \sin x \neq 1$.

$\Rightarrow \sin^2 x - \frac{3}{2}\sin + \frac{1}{2} = 0 \Rightarrow 2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow (2\sin x - 1)(\sin x - 1) = 0$ but $\sin x \neq 1$

$\Rightarrow 2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow \sin x = \sin\frac{\pi}{6}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$.