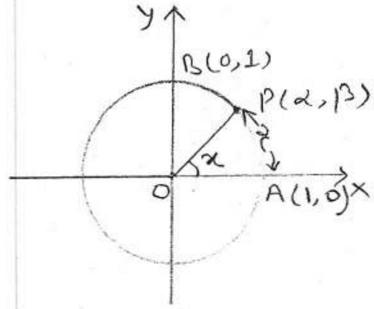
Handwritten notes and problems

by Prof. M. L. Aggarwal

Topic - Trigonometric Functions

Trigonometric functions



1. $\cos x = \alpha$, $\sin x = \beta$

2. **Domain and range of trigonometric functions**

Function	Domain	Range
$\sin x$	R	$[-1,1]$
$\cos x$	R	$[-1,1]$
$\tan x$	$R - \{(2n+1)\frac{\pi}{2}, n \in I\}$	R
$\cot x$	$R - \{n\pi, n \in I\}$	R
$\sec x$	$R - \{(2n+1)\frac{\pi}{2}, n \in I\}$	$(-\infty,-1] \cup [1,\infty)$
$\operatorname{cosec} x$	$R - \{n\pi, n \in I\}$	$(-\infty,-1] \cup [1,\infty)$

3. $\sin(2n\pi + x) = \sin x, \ \forall x \in R$

$\cos(2n\pi + x) = \cos x, \ \forall x \in R$ etc.

3. $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$

$\tan(-x) = -\tan x$, $\cot(-x) = -\cot x$

$\sec(-x) = \sec x$ $\quad \operatorname{cosec}(-x) = -\operatorname{cosec} x$

4. **Identities**

(i) $\sin x \operatorname{cosec} x = 1$, $\cos x \sec x = 1$, $\tan x \cot x = 1$

(ii) $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$

(iii) $\sin^2 x + \cos^2 x = 1$, $1 + \tan^2 x = \sec^2 x$, $1 + \cot^2 x = \operatorname{cosec}^2 x$

(iv) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) = \sin x \cos y - \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\cos(\frac{\pi}{2} - x) = \sin x$, $\sin(\frac{\pi}{2} - x) = \cos x$

$\cos(\frac{\pi}{2} + x) = -\sin x$, $\sin(\frac{\pi}{2} + x) = \cos x$

$\cos(\pi + x) = -\cos x$, $\sin(\pi + x) = -\sin x$

$\cos(\pi - x) = -\cos x$, $\sin(\pi - x) = \sin x$

$\cos(\frac{3\pi}{2} + x) = \sin x$, $\sin(\frac{3\pi}{2} + x) = -\cos x$

etc.

(v) $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$, $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$, $\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

(vi) $2\sin x \cos y = \sin(x+y) + \sin(x-y)$

$2\cos x \sin y = \sin(x+y) - \sin(x-y)$

$2\cos x \cos y = \cos(x+y) + \cos(x-y)$

$2\sin x \sin y = \cos(x-y) - \cos(x+y)$

(vii) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(viii) $\sin 2x = 2\cancel{\sin x\cos x}\ 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(ix) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

(x) $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$

$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

$\tan(x+y+z) = \dfrac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1-\tan x\tan y - \tan y\tan z - \tan z\tan x}$.

(xi) $\sin x\sin(\frac{\pi}{3}+x)\sin(\frac{\pi}{3}-x) = \frac{1}{4}\sin 3x$

$\sin x\sin(\frac{2\pi}{3}+x)(\sin\frac{2\pi}{3}-x) = \frac{1}{4}\sin 3x$

$\cos x\cos(\frac{\pi}{3}+x)\cos(\frac{\pi}{3}-x) = \frac{1}{4}\cos 3x$

$\cos x\cos(\frac{2\pi}{3}+x)\cos(\frac{2\pi}{3}-x) = \frac{1}{4}\cos 3x$

$\tan x\tan(\frac{\pi}{3}+x)\tan(\frac{\pi}{3}-x) = \tan 3x$

$\tan x\tan(\frac{2\pi}{3}+x)\tan(\frac{2\pi}{3}-x) = \tan 3x$

(xii) $\cos\alpha\cos 2\alpha\cos 2^2\alpha\cdots\cos 2^{n-1}\alpha = \dfrac{\sin 2^n\alpha}{2^n\sin\alpha}$

(xiii) $\sin\alpha + \sin(\alpha+\beta) + \cdots + \sin(\alpha+\overline{n-1}\beta) = \dfrac{\sin\frac{n\beta}{2}\sin(\alpha+\frac{n-1}{2}\beta)}{\sin\beta/2}$

$\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+\overline{n-1}\beta) = \dfrac{\sin\frac{n\beta}{2}\cos(\alpha+\frac{n-1}{2}\beta)}{\sin\beta/2}$

(xiv) $\sin 18^\circ = \frac{\sqrt5-1}{4}$, $\cos 36^\circ = \frac{\sqrt5+1}{4}$

$\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt5}}{4}$, $\sin 36^\circ = \dfrac{\sqrt{10-2\sqrt5}}{4}$

$\sin 15^\circ = \dfrac{\sqrt3-1}{2\sqrt2}$, $\cos 15^\circ = \dfrac{\sqrt3+1}{2\sqrt2}$, $\tan 15^\circ = 2-\sqrt3$

$\tan 22\frac{1}{2}^\circ = \sqrt2-1$, $\tan 67\frac{1}{2}^\circ = \sqrt2+1$

(xv) Max. value of $a\cos x + b\sin x = \sqrt{a^2+b^2}$ and minimum value of $a\cos x + b\sin x = -\sqrt{a^2+b^2}$.

1. $\cot 15^\circ + \cot 75^\circ + \cot 135^\circ - \operatorname{cosec} 30^\circ$ is equal to

(a) -1 (b) 0 ✓(c) 1 (d) none of these UD

2. $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ$ is equal to

(a) $\sin 36^\circ$ (b) $\cos 36^\circ$ (c) $\sin 7^\circ$ ✓(d) $\cos 7^\circ$ UD

3. The value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$ is UD

(a) $\frac{1}{2}$ (b) 1 ✓(c) $-\frac{1}{2}$ (d) $\frac{1}{8}$

4. $\sin 6^\circ - \sin 66^\circ + \sin 78^\circ - \sin 42^\circ$ is UD

(a) -1 ✓(b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1

5. The value of $\tan 81^\circ - \tan 63^\circ - \tan 27^\circ + \tan 9^\circ$ equals

(a) 1 (b) 2 (c) 3 ✓(d) 4 UD

6. $\sin 12^\circ \sin 48^\circ \sin 54^\circ$ is equal to

(a) $\frac{1}{4}$ ✓(b) $\frac{1}{8}$ (c) $\frac{1}{2}$ (d) none of these

7. The value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is

(a) $\frac{1}{2}$ (b) $\cos\frac{\pi}{8}$ ✓(c) $\frac{1}{8}$ (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$ UD

8. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then the value of $xy+yz+zx$ is

(a) -1 ✓(b) 0 (c) 1 (d) 2

9. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then $x+y+z$ is equal to

(a) 1 ✓(b) 0 (c) -1 (d) none of these UD

10. If $\frac{x\cos\alpha}{1} = y\cos\left(\alpha + \frac{2\pi}{3}\right) = z\cos\left(\alpha + \frac{4\pi}{3}\right)$, then the value of $xy+yz+zx$ is

(a) 1 ✓(b) 0 (c) -1 (d) none of these

11. If $\sin(y+z-x)$, $\sin(z+x-y)$ and $\sin(x+y-z)$ are in AP then $\tan x$, $\tan y$ and $\tan z$ are in UD

✓(a) AP (b) GP (c) HP (d) none of these

12. If $x_1, x_2, x_3, \ldots, x_n$ are in AP whose common difference is α, then the value of $\sin\alpha\,(\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \cdots + \sec x_{n-1} \sec x_n)$ is

(a) $\frac{\sin(n-1)\alpha}{\cos x_1 \cos x_n}$ (b) $\frac{\sin n\alpha}{\cos x_1 \cos x_n}$ (c) $\sin(n-1)\alpha \cos x_1 \cos x_n$ (d) $\sin\alpha \cos x_1 \cos x_n$

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13. If $\cos(x-y)$, $\cos x$, $\cos(x+y)$ are in HP then $\left|\cos x \sec \frac{y}{2}\right|$ equals

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) none of these UD

14. If $\tan \frac{\pi}{9}$, x, $\tan \frac{5\pi}{18}$ are in AP and $\tan \frac{\pi}{9}$, y, $\tan \frac{7\pi}{18}$ are in AP, then

(a) $2x = y$ (b) $x = y$ (c) $x = 2y$ (d) none of these UD

15. $\sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ$ equals

(a) $\frac{5}{16}$ (b) $\frac{3}{16}$ (c) $\frac{1}{16}$ (d) none of these UD

16. $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8}$ equals

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2 UD

17. The value of $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ}$ equals

(a) 1 (b) 4 (c) 2 (d) 0 UD

18. The value of $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$ is

(a) 0 (b) 1 (c) 2 (d) 4

19. If x, y and z are in AP, then $\frac{\sin x - \sin z}{\cos z - \cos x}$ is equal to

(a) $\tan y$ (b) $\cot y$ (c) $\sin y$ (d) $\cos y$ UD

20. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan \alpha$ is equal to UD

(a) $2(\tan\beta + \tan\gamma)$ (b) $\tan\beta + \tan\gamma$ (c) $\tan\beta + 2\tan\gamma$ (d) $2\tan\beta + \tan\gamma$

21. If $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then $\frac{\tan\alpha}{\tan\beta}$ equals

(a) 1 (b) -1 (c) $\sqrt{2}$ (d) $-\sqrt{2}$ UD

22. If $\cos\theta = \frac{a\cos\phi + b}{a + b\cos\phi}$, then $\tan \frac{\theta}{2}$ equals UD

(a) $\sqrt{\frac{a-b}{a+b}} \tan \frac{\phi}{2}$ (b) $\sqrt{\frac{a+b}{a-b}} \cos \frac{\phi}{2}$ (c) $\sqrt{\frac{a-b}{a+b}} \sin \frac{\phi}{2}$ (d) none of these

23. If $\alpha + \beta = \frac{\pi}{2}$, then the maximum value of $\sin\alpha \sin\beta$ is

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) none of these UD

24. If $f(\theta) = \sin^4\theta + \cos^4\theta + 1$, then the range of $f(\theta)$ is

(a) $\left[\frac{3}{2}, 2\right]$ (b) $\left[1, \frac{3}{2}\right]$ (c) $[1, 2]$ (d) none of these UD

25. The ratio of the greatest value of $2 - \cos x + \sin^2 x$ to its least value is

(a) $\frac{1}{4}$ (b) $\frac{9}{4}$ (c) $\frac{13}{4}$ (d) none of these UD

26. If $f(x) = \frac{\cot x}{1+\cot x}$ and $\alpha + \beta = \frac{5\pi}{4}$, then the value of $f(\alpha) f(\beta)$ is

(a) 2 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) none of these UD

27. If $f(n) = 2\cos nx, \forall n \in \underline{N}$, then $f(1) f(n+1) - f(n)$ is equal to

(a) $f(n+3)$ (b) $f(n+2)$ (c) $f(n+1) f(2)$ (d) $f(n+2) f(2)$ UD.

28. If $4n\alpha = \pi$, then the value of

$\cot\alpha \cot 2\alpha \cot 3\alpha \ldots \cos(2n-1)\alpha$ is

(a) 1 (b) -1 (c) ∞ (d) none of these UD

29. If $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of the equation $8x^2 - 26x + 15 = 0$, then the value of $\cos(\alpha+\beta)$ is

Q.1 276

(a) $-\frac{627}{725}$ (b) $\frac{627}{725}$ (c) 1 (d) none of these UD

30. If $a\cos 2\theta + b\sin 2\theta = c$ has α and β as its solutions, then $\tan\alpha + \tan\beta$ equals

(a) $\frac{2a}{b+c}$ (b) $\frac{2b}{c+a}$ (c) $\frac{2c}{c+b}$ (d) none of these UD

* 31. If $\tan A$, $\tan B$ and $\tan C$ are the roots of the cubic equation $x^3 - 7x^2 + 11x - 7 = 0$ then $A+B+C$ equals

11(iv) 320

(a) $\frac{\pi}{2}$ (b) π (c) 0 (d) none of these UD.

32. The value of $\sum_{r=1}^{7} \tan^2 \frac{r\pi}{16}$ is

(a) 34 (b) 35 (c) 37 (d) none of these UD

33. Given $\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos 2^{n-1}\alpha = \frac{\sin 2^n\alpha}{2^n \sin\alpha}$,

On the basis of above information, answer the following questions:

(i) The value of $\cos\frac{2\pi}{7} \cos\frac{4\pi}{7} \cos\frac{6\pi}{7}$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$ UD

(ii) If $\alpha = \frac{\pi}{13}$, then the value of $\prod_{r=1}^{6} \cos r\alpha$ is

(a) $\frac{1}{64}$ (b) $-\frac{1}{64}$ (c) $\frac{1}{32}$ (d) $-\frac{1}{8}$ UD

(iii) The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{4}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is

(a) 1 (b) $\frac{1}{8}$ (c) $\frac{1}{32}$ (d) $\frac{1}{64}$ UD

(iv) The value of $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$ is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $-\frac{1}{8}$ (d) -1 UD

(v) The value of $64\sqrt{3}\sin\frac{\pi}{48}\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}$ is

(a) 8 (b) 6 (c) 4 (d) -1 UD

34. $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$ equals

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) $\frac{1}{64}$ UD

35. If $\alpha = \frac{\pi}{15}$, then the value of $\prod_{r=1}^{7}\cos r\alpha$ is

(a) $\frac{1}{2^6}$ (b) $\frac{1}{2^7}$ (c) $\frac{1}{2^8}$ (d) none of these

36. The value of $\boxed{\cancel{\sin\frac{\pi}{14}+\sin\frac{3\pi}{14}+\sin\frac{5\pi}{14}}}$ $\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}$ is

(a) $\cot\frac{\pi}{14}$ (b) $\frac{1}{2}\cot\frac{\pi}{14}$ (c) $\tan\frac{\pi}{14}$ (d) $\frac{1}{2}\tan\frac{\pi}{14}$

37. The value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1

38. The value of $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

39. The value of $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+\dots$ to n terms is

(a) 1 (b) 0 (c) $\frac{\pi}{2}$ (d) none of these UD

40. The value of $\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n}$ is UD

(a) $\frac{n}{2}$ (b) $\frac{n}{2}-\frac{1}{2}$ (c) $\frac{n}{2}-1$ (d) none of these

41. If $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ are the roots of the equation $8x^3-4x^2-4x+1=0$, then on the basis of this information, answer the following questions:

(i) The value of $\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}$ is

(a) 2 (b) 4 (c) 8 (d) none of these UD

(ii) The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$ UD

(iii) The value of $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$ UD

42. Read the paragraph and answer the following questions.
If $\alpha, \beta, \gamma, \delta$ are solutions of the equation $\tan(\theta + \frac{\pi}{4}) = 3\tan 3\theta$, no two of which have equal tangents, then

(i) the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$ is UD

(a) $\frac{1}{3}$ (b) $\frac{8}{3}$ (c) $-\frac{8}{3}$ (d) 0

(ii) the value of $\tan\alpha \tan\beta \tan\gamma \tan\delta$ is UD

(a) $-\frac{1}{3}$ (b) -2 (c) 0 (d) none of these

(iii) the value of $\frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \frac{1}{\tan\gamma} + \frac{1}{\tan\delta}$ is UD

(a) -8 (b) 8 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Solutions — Trigonometric functions

1. $\cot 15^\circ + \cot 75^\circ + \cot 135^\circ - \csc 30^\circ = \cot 15^\circ + \tan 15^\circ - 1 - 2$

$= \frac{\cos 15^\circ}{\sin 15^\circ} + \frac{\sin 15^\circ}{\cos 15^\circ} - 3 = \frac{1}{\sin 15^\circ \cos 15^\circ} - 3 = \frac{2}{\sin 30^\circ} - 3 = 2\times 2 - 3 = 1.$

2. $(\sin 47^\circ + \sin 61^\circ) - (\sin 11^\circ + \sin 25^\circ)$

$= 2\sin 54^\circ \cos(-7^\circ) - 2\sin 18^\circ \cos(-7^\circ)$

$= 2\cos 7^\circ (\cos 36^\circ - \sin 18^\circ) = 2\cos 7^\circ \left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = \cos 7^\circ$

3. $(\cos 12^\circ + \cos 84^\circ) - (\cos 24^\circ + \cos 48^\circ)$

$= 2\cos 48^\circ \cos 36^\circ - 2\cos 36^\circ \cos 12^\circ = 2\cos 36^\circ (\cos 48^\circ - \cos 12^\circ)$

$= 2\cos 36^\circ \cdot 2\sin 30^\circ \sin(-18^\circ) = -4 \cdot \frac{1}{2} \cdot \frac{\sqrt{5}+1}{4} \cdot \frac{\sqrt{5}-1}{4} = -\frac{1}{2}$

4. $(\sin 6^\circ - \sin 66^\circ) + (\sin 78^\circ - \sin 42^\circ)$

$= -2\cos 36^\circ \sin 30^\circ + 2\cos 60^\circ \sin(18^\circ)$

$= -2 \cdot \frac{1}{2}\cos 36^\circ + 2\cdot\frac{1}{2}\sin 18^\circ = -\frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4} = -\frac{1}{2}$

5. $(\tan 81^\circ + \tan 9^\circ) - (\tan 63^\circ + \tan 27^\circ)$

$= (\cot 9^\circ + \tan 9^\circ) - (\cot 27^\circ + \tan 27^\circ)$

$= \frac{\cos^2 9^\circ + \sin^2 9^\circ}{\sin 9^\circ \cos 9^\circ} - \frac{\cos^2 27^\circ + \sin^2 27^\circ}{\sin 27^\circ \cos 27^\circ} = \frac{1}{\sin 9^\circ \cos 9^\circ} - \frac{1}{\sin 27^\circ \cos 27^\circ}$

$= \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} = \frac{2}{\sin 18^\circ} - \frac{2}{\cos 36^\circ} = \frac{2\times 4}{\sqrt{5}-1} - \frac{2\times 4}{\sqrt{5}+1}$

$= 8 \frac{(\sqrt{5}+1) - (\sqrt{5}-1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = 8 \times \frac{2}{4} = 4$

6. $\frac{1}{2}(2\sin 12^\circ \sin 48^\circ)\sin(90^\circ - 36^\circ) = \frac{1}{2}(\cos 36^\circ - \cos 60^\circ)\cos 36^\circ$

$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right] \times \frac{\sqrt{5}+1}{4} = \frac{1}{2} \cdot \frac{\sqrt{5}-1}{4} \times \frac{\sqrt{5}+1}{4} = \frac{1}{8}$

7. $\cos\frac{7\pi}{8} = \cos(\pi - \frac{\pi}{8}) = -\cos\frac{\pi}{8}$, $\cos\frac{5\pi}{8} = \cos(\pi - \frac{3\pi}{8}) = -\cos\frac{3\pi}{8}$.

Given expression $= (1+\cos\frac{\pi}{8})(1+\cos\frac{3\pi}{8})(1-\cos\frac{3\pi}{8})(1-\cos\frac{\pi}{8})$

$= (1-\cos^2\frac{\pi}{8})(1-\cos^2\frac{3\pi}{8}) = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$, $\sin\frac{3\pi}{8} = \sin(\frac{\pi}{2} - \frac{\pi}{8}) = \cos\frac{\pi}{8}$

$= \sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8} = \frac{(2\sin\frac{\pi}{8}\cos\frac{\pi}{8})^2}{4} = \frac{\sin^2\frac{\pi}{4}}{4} = \frac{\frac{1}{2}}{4} = \frac{1}{8}$

8. $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3} \Rightarrow x = -\frac{y}{2} = -\frac{z}{2} = \lambda$ (say)

$\Rightarrow x = \lambda,\ y = -2\lambda,\ z = -2\lambda.$

$\therefore xy + yz + zx = \lambda(-2\lambda) + (-2\lambda)(-2\lambda) + (-2\lambda)\lambda = 0$

9. $\frac{x}{\cos\theta} = \frac{y}{\cos(\theta - \frac{2\pi}{3})} = \frac{z}{\cos(\theta + \frac{2\pi}{3})} = k$ (say) $\Rightarrow x = k\cos\theta$ etc.

$\therefore\ x+y+z = h\left[\cos\theta + \cos\left(\theta-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)\right]$ $\left[\cos x+\cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\right]$

$= h\left[\cos\theta + 2\cos\theta\cos\left(-\frac{2\pi}{3}\right)\right]$

$= h\left[\cos\theta + 2\cos\theta\cos\frac{2\pi}{3}\right] = h\left[\cos\theta + 2\cos\theta\left(-\frac{1}{2}\right)\right] = 0$

10. $x\cos\alpha = y\cos\left(\alpha+\frac{2\pi}{3}\right) = z\cos\left(\alpha+\frac{4\pi}{3}\right) = h$ (say)

$\Rightarrow x = \frac{h}{\cos\alpha},\ y = \frac{h}{\cos\left(\alpha+\frac{2\pi}{3}\right)},\ z = \frac{h}{\cos\left(\alpha+\frac{4\pi}{3}\right)}$

$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{h}\left[\cos\alpha + \cos\left(\alpha+\frac{2\pi}{3}\right)+\cos\left(\alpha+\frac{4\pi}{3}\right)\right]$

$= \frac{1}{h}\left[\cos\alpha + 2\cos(\alpha+\pi)\cos\left(-\frac{\pi}{3}\right)\right] = \frac{1}{h}\left[\cos\alpha - 2\cos\alpha\cos\frac{\pi}{3}\right]$

$= \frac{1}{h}\left[\cos\alpha - 2\cos\alpha\cdot\frac{1}{2}\right] = 0$

$\Rightarrow \frac{xy+yz+zx}{xyz} = 0 \Rightarrow xy+yz+zx = 0.$

11. $\sin(z+x-y) - \sin(y+z-x) = \sin(x+y-z) - \sin(z+x-y)$

$\Rightarrow 2\cos z\sin(x-y) = 2\cos x\sin(y-z)$

$\Rightarrow \cos z\sin x\cos y - \cos z\cos x\sin y = \cos x\sin y\cos z - \cos x\cos y\sin z$

(Divide both sides by $\cos x\cos y\cos z$)

$\Rightarrow \tan x - \tan y = \tan y - \tan z \Rightarrow \tan x + \tan z = 2\tan y$

$\Rightarrow \tan x, \tan y$ and $\tan z$ are in AP.

12. Given expression $= \frac{\sin\alpha}{\cos x_1\cos x_2} + \frac{\sin\alpha}{\cos x_2\cos x_3} + \dots + \frac{\sin\alpha}{\cos x_{n-1}\cos x_n}$

$= \frac{\sin(x_2-x_1)}{\cos x_1\cos x_2} + \frac{\sin(x_3-x_2)}{\cos x_2\cos x_3} + \dots + \frac{\sin(x_n-x_{n-1})}{\cos x_{n-1}\cos x_n}$

$= \frac{\sin x_2\cos x_1 - \cos x_2\sin x_1}{\cos x_1\cos x_2} + \dots$

$= (\tan x_2 - \tan x_1) + (\tan x_3 - \tan x_2) + \dots + (\tan x_n - \tan x_{n-1})$

$= \tan x_n - \tan x_1 = \frac{\sin x_n}{\cos x_n} - \frac{\sin x_1}{\cos x_1} = \frac{\sin x_n\cos x_1 - \cos x_n\sin x_1}{\cos x_1\cos x_n}$

$= \frac{\sin(x_n-x_1)}{\cos x_1\cos x_n} = \frac{\sin(n-1)\alpha}{\cos x_1\cos x_n}$ $(\because x_n = x_1+(n-1)\alpha)$

13. a, b, c are in HP $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP $\Rightarrow \frac{2}{b} = \frac{1}{a}+\frac{1}{c} = \frac{a+c}{ac} \Rightarrow b = \frac{2ac}{a+c}$

$\cos(x-y), \cos x, \cos(x+y)$ are in HP

$\Rightarrow \cos x = \frac{2\cos(x-y)\cos(x+y)}{\cos(x+y)+\cos(x-y)} \Rightarrow \cos x = \frac{2(\cos^2x - \sin^2y)}{2\cos x\cos y}$

$\Rightarrow \cos^2x\cos y = \cos^2x - \sin^2y \Rightarrow \sin^2y = \cos^2x(1-\cos y)$

$\left(2\sin\frac{y}{2}\cos\frac{y}{2}\right)^2 = \cos^2x\cdot 2\sin^2\frac{y}{2} \Rightarrow 2\cos^2\frac{y}{2} = \cos^2x \Rightarrow \cos x\sec\frac{y}{2} = \pm\sqrt{2}$

14. $\tan\frac{\pi}{9}, x, \tan\frac{5\pi}{18}$ are in AP $\Rightarrow 2x = \tan 20° + \tan 50°$

$$\Rightarrow 2x = \frac{\sin 20°}{\cos 20°} + \frac{\sin 50°}{\cos 50°} = \frac{\sin 20°\cos 50° + \cos 20°\sin 50°}{\cos 20°\cos 50°} = \frac{\sin 70°}{\cos 20°\cos 50°}$$

$$= \frac{\cos 20°}{\cos 20°\sin 40°} = \frac{1}{\sin 40°} \quad \text{---(i)}$$

$\tan\frac{\pi}{9}, y, \tan\frac{7\pi}{18}$ are in AP $\Rightarrow 2y = \tan 20° + \tan 70°$

$$\Rightarrow 2y = \frac{\sin 20°\cos 70° + \cos 20°\sin 70°}{\cos 20°\cos 70°} = \frac{\sin 90°}{\cos 20°\sin 20°} = \frac{1.2}{2\sin 20°\cos 20°} = \frac{2}{\sin 40°}$$

$$\Rightarrow y = \frac{1}{\sin 40°} \quad \text{---(ii)}$$

from (i) & (ii) $\quad 2x = y$.

15. $\sin 36°\sin 72°\sin 108°\sin 144° = \sin 36°\cos 18°\cos 18°\sin 36°$

$$= \sin^2 36°\cos^2 18° = \frac{10-2\sqrt{5}}{16}\cdot\frac{10+2\sqrt{5}}{16} = \frac{100-20}{256} = \frac{80}{256} = \frac{5}{16}$$

16. $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$; $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

$\therefore$ Given expression $= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \left(-\cos\frac{3\pi}{8}\right)^4 + \left(-\cos\frac{\pi}{8}\right)^4$

$$= 2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right) \qquad \Big|\ \cos\frac{3\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin\frac{\pi}{8}$$

$$= 2\left(\cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8}\right) = 2\left[\left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right)^2 - 2\cos^2\frac{\pi}{8}\sin^2\frac{\pi}{8}\right]$$

$$= 2\left[1 - 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right] = 2 - \left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2$$

$$= 2 - \sin^2\frac{\pi}{4} = 2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

17. $$\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = \frac{\cos 10° - \sqrt{3}\sin 10°}{\sin 10°\cos 10°} = 4\cdot\frac{\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°}{2\sin 10°\cos 10°}$$

$$= 4\cdot\frac{\cos 60°\cos 10° - \sin 60°\sin 10°}{\sin 20°} = 4\,\frac{\cos 70°}{\sin 20°} = 4\cdot\frac{\sin 20°}{\sin 20°} = 4$$

18. Given express $= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°} = 4\cdot\frac{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°}{2\sin 20°\cos 20°}$

$$= 4\cdot\frac{\cos 30°\cos 20° - \sin 30°\sin 20°}{\sin 40°} = 4\,\frac{\cos 50°}{\sin 40°} = 4\cdot 1.$$

19. x, y, z are in AP $\Rightarrow x + z = 2y$.

$$\frac{\sin x - \sin z}{\cos z - \cos x} = \frac{2\cos\frac{x+z}{2}\sin\frac{x-z}{2}}{2\sin\frac{z+x}{2}\sin\frac{x-z}{2}} = \cot\frac{x+z}{2} = \cot\frac{2y}{2} = \cot y.$$

20. $\beta + \gamma = \alpha \Rightarrow \gamma = \alpha - \beta$ $\quad\quad$ $\alpha + \beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \alpha$

$\Rightarrow \tan\gamma = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ $\quad\quad$ $\Rightarrow \tan\beta = \tan(\frac{\pi}{2} - \alpha) = \cot\alpha$

$\Rightarrow \tan\gamma = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\cot\alpha} = \frac{\tan\alpha - \tan\beta}{1+1} \Rightarrow 2\tan\gamma = \tan\alpha - \tan\beta$

$\Rightarrow \tan\alpha = \tan\beta + 2\tan\gamma$

21. $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta} \Rightarrow \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{3 \cdot \frac{1 - \tan^2\beta}{1 + \tan^2\beta} - 1}{3 - \frac{1 - \tan^2\beta}{1 + \tan^2\beta}} = \frac{2 - 4\tan^2\beta}{2 + \tan^2\beta}$

$\Rightarrow \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{1 - 2\tan^2\beta}{1 + 2\tan^2\beta}$ $\quad$ (Apply componendo dividendo)

$\Rightarrow \frac{2}{2\tan^2\alpha} = \frac{2}{4\tan^2\beta} \Rightarrow \frac{1}{\tan^2\alpha} = \frac{1}{2\tan^2\beta} \Rightarrow \frac{\tan^2\alpha}{\tan^2\beta} = 2 \Rightarrow \frac{\tan\alpha}{\tan\beta} = \pm\sqrt{2}$

22. $\cos\theta = \frac{a\cos\phi + b}{a + b\cos\phi} \Rightarrow \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{a\frac{1 - \tan^2\phi/2}{1 + \tan^2\phi/2} + b}{a + b \cdot \frac{1 - \tan^2\phi/2}{1 + \tan^2\phi/2}}$

$\Rightarrow \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{(a+b) - (a-b)\tan^2\frac{\phi}{2}}{(a+b) + (a-b)\tan^2\frac{\phi}{2}}$ $\quad$ (App. compo & dividendo)

$\Rightarrow \frac{2}{2\tan^2\frac{\theta}{2}} = \frac{2(a+b)}{2(a-b)\tan^2\frac{\phi}{2}} \Rightarrow \tan^2\frac{\theta}{2} = \frac{a-b}{a+b}\tan^2\frac{\phi}{2} \Rightarrow \tan\frac{\theta}{2} = \pm\sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$

23. $\sin\alpha\sin\beta = \sin\alpha\sin(\frac{\pi}{2} - \alpha) = \sin\alpha\cos\alpha = \frac{\sin 2\alpha}{2}$

As $-1 \le \sin 2\alpha \le 1$ for all $\alpha \in R$ $\Rightarrow -\frac{1}{2} \le \frac{\sin 2\alpha}{2} \le \frac{1}{2}$

~~$\therefore$ Max. value of $\sin\alpha\sin\beta$ $\frac{\sin 2\alpha}{2}$ is $\frac{1}{2}$~~.

$\therefore$ Maximum value of $\sin\alpha\sin\beta$ i.e. $\frac{\sin 2\alpha}{2}$ is $\frac{1}{2}$

24. $f(\theta) = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta + 1 = 1 - \frac{1}{2}(2\sin\theta\cos\theta)^2 + 1$

$= 2 - \frac{1}{2}\sin^2 2\theta$ $\quad\quad$ $0 \le \sin^2 2\theta \le 1.$

Min. value of $f(\theta) = 2 - \frac{1}{2} \cdot 1 = \frac{3}{2}$, $\quad\quad$ $\therefore$ Range $= [\frac{3}{2}, 2]$

Max. value of $f(\theta) = 2 - \frac{1}{2} \cdot 0 = 2$.

25. $2 - \cos x + \sin^2 x = 2 - \cos x + (1 - \cos^2 x) = -(\cos^2 x + \cos x) + 3$

$= -(\cos x + \frac{1}{2})^2 + \frac{1}{4} + 3 = \frac{13}{4} - (\cos x + \frac{1}{2})^2$

Max. value $= \frac{13}{4} - 0$ (when $\cos x = -\frac{1}{2}$) $= \frac{13}{4}$.

Min. value $= \frac{13}{4} - (1 + \frac{1}{2})^2$ (when $\cos x = 1$)

$= \frac{13}{4} - \frac{9}{4} = 1$

$\therefore$ Ratio of max. value : min. value $= \frac{13}{4} : 1 = \frac{13}{4}$

26. $f(\alpha)f(\beta) = \frac{\cot\alpha}{1 + \cot\alpha} \cdot \frac{\cot\beta}{1 + \cot\beta} = \frac{1}{1 + \tan\alpha} \cdot \frac{1}{1 + \tan\beta} = \frac{1}{1 + \tan\alpha} \cdot \frac{1}{1 + \tan(\frac{5\pi}{4} - \alpha)}$

$\tan(\frac{5\pi}{4} - \alpha) = \tan(\pi + \frac{\pi}{4} - \alpha) = \tan(\frac{\pi}{4} - \alpha)$

$= \frac{1}{1 + \tan\alpha} \cdot \frac{1}{1 + \tan(\frac{\pi}{4} - \alpha)} = \frac{1}{1 + \tan\alpha} \cdot \frac{1}{1 + \frac{1 - \tan\alpha}{1 + \tan\alpha}}$

$$= \frac{1}{1+|\tan\alpha|} \cdot \frac{1+|\tan\alpha|}{2} = \frac{1}{2}$$

27. Given $f(n) = 2\cos nx, \ \forall\, n \in N$

$\Rightarrow f(1) = 2\cos x, \quad f(n+1) = 2\cos(n+1)x$

$\therefore f(1)\,f(n+1) - f(n) = 2\cos x \cdot 2\cos(n+1)x - 2\cos nx$

$= 2\left[\underline{2\cos(n+1)x\cos x} - \cos nx\right]$

$= 2\left[\cos(n+2)x + \cos nx - \cos nx\right] = 2\cos(n+2)x = f(n+2)$

28. $\cot\alpha \cot(2n-1)\alpha = \cot\alpha \cot(2n\alpha - \alpha) = \cot\alpha \cot(\frac{\pi}{2} - \alpha)$

$= \cot\alpha \tan\alpha = 1$ etc.

So, the product of terms equidistant from beginning and end are equal.

Total no. of terms $= 2n-1$, which is odd.

So, there is one middle term.

Middle term $= \cot n\alpha = \cot\frac{\pi}{4} = 1$.

$\therefore$ Given expression $= 1.1.1\ldots n$ times $= 1$

29. $\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \dfrac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \dfrac{\frac{26}{8}}{1 - \frac{15}{8}} = -\dfrac{26}{7}$

$8x^2 - 26x + 15 = 0$
roots are $\tan\frac{\alpha}{2}, \tan\frac{\beta}{2}$

$$\cos(\alpha+\beta) = \frac{1 - \tan^2\frac{\alpha+\beta}{2}}{1 + \tan^2\frac{\alpha+\beta}{2}} = \frac{1 - (-\frac{26}{7})^2}{1 + (-\frac{26}{7})^2} = \frac{1 - \frac{676}{49}}{1 + \frac{676}{49}} = -\frac{627}{725}$$

30. $a\cos 2\theta + b\sin 2\theta = c \Rightarrow a \cdot \dfrac{1-\tan^2\theta}{1+\tan^2\theta} + b \cdot \dfrac{2\tan\theta}{1+\tan^2\theta} = c$

$\Rightarrow a(1-\tan^2\theta) + 2b\tan\theta = c(1+\tan^2\theta)$

$\Rightarrow (c+a)\tan^2\theta - 2b\tan\theta + (c-a) = 0$; α and β are values of $\underline{\theta}$.

$\therefore \tan\alpha + \tan\beta = -\dfrac{-2b}{c+a} = \dfrac{2b}{c+a}$.

31. $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

$x^3 - 7x^2 + 11x - 7 = 0$
$\Sigma \tan A = 7$,
$\Sigma \tan A\tan B = 11$
$\tan A\tan B\tan C = 7$

$$= \frac{7-7}{1-11} = \frac{0}{-10} = 0$$

$\Rightarrow A+B+C = n\pi, \ n \in N$

$\Rightarrow A+B+C = 0, \pi$

32. $\tan\frac{7\pi}{16} = \tan\left(\frac{\pi}{2} - \frac{\pi}{16}\right) = \cot\frac{\pi}{16}$, $\tan\frac{6\pi}{16} = \tan\left(\frac{\pi}{2} - \frac{2\pi}{16}\right) = \cot\frac{2\pi}{16}$,

$\tan\frac{5\pi}{16} = \tan\left(\frac{\pi}{2} - \frac{3\pi}{16}\right) = \cot\frac{3\pi}{16}$; and $\tan\frac{4\pi}{16} = \tan\frac{\pi}{4} = 1$.

$$\therefore \sum_{r=1}^{7} \tan^2\frac{r\pi}{16} = \left(\tan^2\frac{\pi}{16} + \cot^2\frac{\pi}{16}\right) + \left(\tan^2\frac{2\pi}{16} + \cot^2\frac{2\pi}{16}\right) + \left(\tan^2\frac{3\pi}{16} + \cot^2\frac{3\pi}{16}\right) + 1$$

Now $\tan^2 x + \cot^2 x = \dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{\sin^4 x + \cos^4 x}{\sin^2 x\cos^2 x} = \dfrac{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}{\sin^2 x \cdot \cos^2 x}$

$$= \frac{1-2\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x} = \frac{1}{\sin^2 x\cos^2 x} - 2 = \frac{4}{(2\sin x\cos x)^2} - 2 = \frac{4}{\sin^2 2x} - 2$$

$$= \frac{4}{\frac{1-\cos 4x}{2}} - 2 = \frac{8}{1-\cos 4x} - 2 .$$

$$\therefore \sum_{r=1}^{7} \tan^2 \frac{r\pi}{16} = \left(\frac{8}{1-\cos\frac{\pi}{4}} - 2\right) + \left(\frac{8}{1-\cos\frac{\pi}{2}} - 2\right) + \left(\frac{8}{1-\cos\frac{3\pi}{4}} - 2\right) + 1$$

$$= \frac{8}{1-\frac{1}{\sqrt2}} + \frac{8}{1-0} + \frac{8}{1+\frac{1}{\sqrt2}} - 6 + 1$$

$$= 8\left[\frac{\sqrt2}{\sqrt2-1} + \frac{\sqrt2}{\sqrt2+1}\right] + 8 - 6 + 1 = 8\sqrt2 . \frac{(\sqrt2+1)+\sqrt2-1}{(\sqrt2-1)(\sqrt2+1)} + 3$$

$$= 8\sqrt2 . \frac{2\sqrt2}{2-1} + 3 = \frac{32}{1} + 3 = 35 .$$

33. (i) $\cos\frac{6\pi}{7} = \cos(\pi - \frac{\pi}{7}) = -\cos\frac{\pi}{7}$

Given expression $= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}(-\cos\frac{\pi}{7}) = -\cos\alpha\cos2\alpha\cos4\alpha$, when $\alpha = \frac{\pi}{7}$

$$= -\frac{\sin 8\alpha}{8\sin\alpha} = -\frac{\sin(7\alpha+\alpha)}{8\sin\alpha} = -\frac{\sin(\pi+\alpha)}{8\sin\alpha} = \frac{1}{8}$$

(ii) $\alpha = \frac{\pi}{13} \Rightarrow 13\alpha = \pi$

$$\prod_{r=1}^{6} \cos r\alpha = \cos\alpha\cos2\alpha\cos3\alpha\cos4\alpha\cos5\alpha\cos6\alpha$$

$$(\cos 5\alpha = \cos(13\alpha - 8\alpha) = \cos(\pi - 8\alpha) = -\cos 8\alpha)$$

$$= -(\cos\alpha\cos2\alpha\cos4\alpha\cos8\alpha)(\cos3\alpha\cos6\alpha)$$

$$= -\frac{\sin16\alpha}{16\sin\alpha} . \cancel{\frac{\sin 12\alpha}{4\sin\alpha}} (\cos3\alpha\cos6\alpha)$$

$$(\sin16\alpha = \sin(13\alpha+3\alpha) = \sin(\pi+3\alpha)) = -\sin3\alpha$$

$$= -\frac{-\sin3\alpha}{16\sin\alpha} . \cos3\alpha\cos6\alpha = \frac{1}{16}\cdot\frac{1}{2}\frac{\sin6\alpha\cos6\alpha}{\sin\alpha}$$

$$= \frac{1}{32}\cdot\frac{1}{2}\frac{\sin12\alpha}{\sin\alpha} = \frac{1}{64}\frac{\sin(13\alpha-\alpha)}{\sin\alpha} = \frac{1}{64}\frac{\sin(\pi-\alpha)}{\sin\alpha} = \frac{1}{64}$$

(iii) $\sin\frac{13\pi}{14} = \sin(\pi - \frac{\pi}{14}) = \sin\frac{\pi}{14}$; $\sin\frac{11\pi}{14} = \sin(\pi - \frac{3\pi}{14}) = \sin\frac{3\pi}{14}$;
$\sin\frac{9\pi}{14} = \sin(\pi - \frac{5\pi}{14}) = \sin\frac{5\pi}{4}$; $\sin\frac{7\pi}{14} = \sin\frac{\pi}{2} = 1$.

$\therefore$ Given expression $= \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{4}\sin\frac{5\pi}{14}\right)^2 . 1$

$$= \left(\cos(\tfrac{\pi}{2}-\tfrac{\pi}{14})\cos(\tfrac{\pi}{2}-\tfrac{3\pi}{14})\cos(\tfrac{\pi}{2}-\tfrac{5\pi}{14})\right)^2 = \left(\cos\frac{6\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}\right)^2$$

$$= \left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)^2$$

$\cos\frac{3\pi}{7} = (\cos\pi - \frac{4\pi}{7}) = -\cos\frac{4\pi}{7}$
$\alpha = \frac{\pi}{7}$ i.e. $7\alpha = \pi$

$$= \left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\right)^2 = (\cos\alpha\cos2\alpha\cos4\alpha)^2 = \left(\frac{\sin8\alpha}{8\sin\alpha}\right)^2$$

$$= \frac{1}{64} \quad [\sin8\alpha = \sin(7\alpha+\alpha) = \sin(\pi+\alpha) = -\sin\alpha]$$

(iv) $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18} = \cos(\frac{\pi}{2}-\frac{\pi}{18})\cos(\frac{\pi}{2}-\frac{5\pi}{18})\cos(\frac{\pi}{2}-\frac{7\pi}{18})$

$= \cos\frac{8\pi}{18}\cos\frac{4\pi}{18}\cos\frac{2\pi}{18} = \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \cos\alpha\cos2\alpha\cos4\alpha,\ \alpha=\frac{\pi}{9}$

$= \frac{\sin 8\alpha}{8\sin\alpha} = \frac{\sin(9\alpha-\alpha)}{8\sin\alpha} = \frac{\sin(\pi-\alpha)}{8\sin\alpha} = \frac{1}{8}.$

(v) Let $\frac{\pi}{48}=\alpha \Rightarrow 48\alpha=\pi$

Given expression $= 64\sqrt{3}\,\underline{\sin\alpha\cos\alpha}\cos2\alpha\cos4\alpha\cos8\alpha$

$= 32\sqrt{3}\,\underline{\sin2\alpha\cos2\alpha}\cos4\alpha\cos8\alpha$

$= 16\sqrt{3}\,\underline{\sin4\alpha\cos4\alpha}\cos8\alpha$

$= 8\sqrt{3}\,\underline{\sin8\alpha\cos8\alpha} = 4\sqrt{3}\sin16\alpha$

$= 4\sqrt{3}\sin\frac{\pi}{3} = 4.\sqrt{3}.\frac{\sqrt{3}}{2} = 6.$

34. Let $\frac{\pi}{65}=\alpha$; given expression $= \cos\alpha\cos2\alpha\cos4\alpha\cos8\alpha\cos16\alpha\cos32\alpha$

$= \frac{\sin 64\alpha}{64\sin\alpha} = \frac{\sin(65\alpha-\alpha)}{64\sin\alpha} = \frac{\sin(\pi-\alpha)}{64\sin\alpha} = \frac{1}{64}$

35. Given expression $= \cos\alpha\cos2\alpha\cos3\alpha\cos4\alpha\cos5\alpha\cos6\alpha\cos7\alpha$

$[\cos7\alpha = \cos(15\alpha-8\alpha) = \cos(\pi-8\alpha) = -\cos8\alpha]$

$= -(\cos\alpha\cos2\alpha\cos4\alpha\cos8\alpha)(\cos3\alpha\cos6\alpha).\cos5\alpha$

$= -\frac{\sin16\alpha}{16\sin\alpha}.\frac{\sin12\alpha}{4\sin3\alpha}.\cos\frac{\pi}{3} \quad (5\alpha=\frac{\pi}{3})$

$= -\frac{\sin(15\alpha+\alpha)}{64\sin\alpha}.\frac{\sin(15\alpha-3\alpha)}{4\sin3\alpha}.\frac{1}{2} \qquad \underline{15\alpha=\pi.}$

$= -\frac{-\sin\alpha}{64\sin\alpha}.\frac{\sin3\alpha}{\sin3\alpha}.\frac{1}{2} = \frac{1}{128} = \frac{1}{2^7}.$

36. Use $\sin\alpha+\sin(\alpha+\beta)+\dots$ to n terms $= \frac{\sin n\frac{\beta}{2}\sin(\alpha+(n-1)\frac{\beta}{2})}{\sin\frac{\beta}{2}}$

Here $\alpha=\frac{\pi}{7},\ \beta=\frac{\pi}{7} \Rightarrow \frac{\beta}{2}=\frac{\pi}{14},\ n=3.$

Required sum $= \frac{\sin\frac{3\pi}{14}\sin(\frac{\pi}{7}+(3-1)\frac{\pi}{14})}{\sin\frac{\pi}{14}} = \frac{2}{2}.\frac{\sin\frac{3\pi}{14}\sin\frac{4\pi}{14}}{\sin\frac{\pi}{14}}$

$= \frac{\cos\frac{\pi}{14}-\cos\frac{7\pi}{14}}{2\sin\frac{\pi}{14}} = \frac{\cos\frac{\pi}{14}-0}{2\sin\frac{\pi}{14}} = \frac{1}{2}\cot\frac{\pi}{14}$

37. $\alpha=\frac{2\pi}{7},\ \frac{\beta}{2}=\frac{\pi}{7},\ n=3$

Sum $= \frac{\sin\frac{3\pi}{7}\cos(2\frac{\pi}{7}+2.\frac{\pi}{7})}{\sin\frac{\pi}{7}} = \frac{\sin\frac{3\pi}{7}\cos\frac{4\pi}{7}}{\sin\frac{\pi}{7}} = -\frac{\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$ $\qquad \cos\frac{4\pi}{7}=\cos(\pi-\frac{3\pi}{7})$

$= -\frac{1}{2}\frac{\sin 6\pi/7}{\sin\pi/7} = -\frac{1}{2}\frac{\sin(\pi-\pi/7)}{\sin\pi/7} = -\frac{1}{2}$

38. $\alpha = \frac{\pi}{11}$, $\beta = \frac{2\pi}{11} \Rightarrow \frac{\beta}{2} = \frac{\pi}{11}$, $n = 5$

$$\text{Sum} = \frac{\sin 5\frac{\pi}{11}}{\sin\frac{\pi}{11}} \cos\left(\frac{\pi}{11} + 4\cdot\frac{\pi}{11}\right) = \frac{\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}}{\sin\frac{\pi}{11}} = \frac{\sin\frac{10\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{1}{2}.$$

39. $\alpha = \frac{\pi}{n}$, $\frac{\beta}{2} = \frac{\pi}{n}$, no. of terms is n.

$$\text{Sum} = \frac{\sin\left(n\cdot\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}+(n-1)\frac{\pi}{n}\right)}{\sin\frac{\pi}{n}} = \frac{\sin\pi\,\sin\pi}{\sin\frac{\pi}{n}} = 0$$

40. $$\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n} = \sum_{r=1}^{n-1}\frac{1+\cos 2\frac{r\pi}{n}}{2} = \frac{1}{2}\sum_{r=1}^{n-1} 1 + \frac{1}{2}\sum_{r=1}^{n-1}\cos\frac{2r\pi}{n}$$

$$= \frac{n-1}{2} + \frac{1}{2}\left(\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \dots \text{ to } (n-1) \text{ terms}\right)$$

$$\left(\alpha = \frac{2\pi}{n},\ \beta = \frac{2\pi}{n} \Rightarrow \frac{\beta}{2} = \frac{\pi}{n}\right)$$

$$= \frac{n-1}{2} + \frac{1}{2}\cdot\frac{\sin(n-1)\frac{\pi}{n}\,\cos\left(\frac{2\pi}{n} + (n-2)\frac{\pi}{n}\right)}{\sin\frac{\pi}{n}}$$

$$\left[\sin\left(\pi - \frac{\pi}{n}\right) = \sin\frac{\pi}{n};\ \cos\left(\frac{2\pi}{n}+\pi-\frac{2\pi}{n}\right) = \cos\pi = -1\right]$$

$$= \frac{n-1}{2} + \frac{1}{2}(-1) = \frac{n}{2} - \frac{1}{2} - \frac{1}{2} = \frac{n}{2} - 1.$$

41. Let $\frac{\pi}{7}=\alpha$, $\frac{3\pi}{7}=\beta$ and $\frac{5\pi}{7}=\gamma$ then $\cos\alpha, \cos\beta, \cos\gamma$ are roots of $8x^3-4x^2-4x+1=0$

$\cos\alpha+\cos\beta+\cos\gamma = \frac{1}{2}$, $\Sigma\cos\alpha\cos\beta = -\frac{1}{2}$, $\cos\alpha\cos\beta\cos\gamma = -\frac{1}{8}$

(i) $$\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7} = \frac{1}{\cos\alpha}+\frac{1}{\cos\beta}+\frac{1}{\cos\gamma} = \frac{\cos\beta\cos\gamma+\dots}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \frac{-\frac{1}{2}}{-\frac{1}{8}} = \frac{1}{2}\times\frac{8}{1} = 4.$$

(ii) $$\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2 = \frac{1-\cos\alpha}{2}\cdot\frac{1-\cos\beta}{2}\cdot\frac{1-\cos\gamma}{2}$$

$$= \frac{1}{8}\left[1^3 - (\cos\alpha+\cos\beta+\cos\gamma)\cdot 1^2 + \Sigma\cos\alpha\cos\beta\cdot 1 - \cos\alpha\cos\beta\cos\gamma\right]$$

$$= \frac{1}{8}\left[1-\frac{1}{2}-\frac{1}{2}+\frac{1}{8}\right] = \frac{1}{64} \Rightarrow \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}.$$

(iii) $$\left(\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}\right)^2 = \frac{1+\cos\alpha}{2}\cdot\frac{1+\cos\beta}{2}\cdot\frac{1+\cos\gamma}{2}$$

$$= \frac{1}{8}\left[1+\Sigma\cos\alpha+\Sigma\cos\alpha\cos\beta+\cos\alpha\cos\beta\cos\gamma\right]$$

$$= \frac{1}{8}\left[1+\frac{1}{2}+\left(-\frac{1}{2}\right)+\left(-\frac{1}{8}\right)\right] = \frac{1}{8}\cdot\frac{7}{8} = \frac{7}{64}$$

$$\Rightarrow \cos\frac{\pi}{4}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$$

42 $\tan(\theta+\frac{\pi}{4}) = 3\tan 3\theta \Rightarrow \dfrac{\tan\theta+1}{1-\tan\theta} = 3.\ \dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

$\Rightarrow 9\tan\theta - 3\cancel{\tan^2\theta} - 9\tan^3\theta + 3\tan^4\theta = \tan\theta - 3\cancel{\tan^2\theta} + 1 - 3\tan^2\theta$

$\Rightarrow 3\tan^4\theta + 0.\tan^3\theta - 6\tan^2\theta + 8\tan\theta - 1 = 0.$

As $\alpha, \beta, \gamma, \delta$ are its solutions

$\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = -\frac{0}{3} = 0 \quad \ldots (i)$

$\tan\alpha\tan\beta\tan\gamma\tan\delta = +\frac{-1}{3} = -\frac{1}{3} \quad \ldots (ii)$

$\Sigma\tan\alpha\tan\beta\tan\gamma = -\frac{8}{3}$

$\frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \frac{1}{\tan\gamma} + \frac{1}{\tan\delta} = \dfrac{\Sigma\tan\alpha\tan\beta\tan\gamma}{\tan\alpha\tan\beta\tan\gamma\tan\delta}$

$= \dfrac{-\frac{8}{3}}{-\frac{1}{3}} = 8$